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Marie-Françoise Bidaut-Véron

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# Local and global estimates of solutions of Hamilton-Jacobi parabolic equation with absorption

Marie Françoise BIDAUT-VERON

## Abstract

Here we show new apriori estimates for the nonnegative solutions of the equation

$$u_t - \Delta u + |\nabla u|^q = 0$$

in  $Q_{\Omega,T} = \Omega \times (0, T)$ ,  $T \leq \infty$ , where  $q > 0$ , and  $\Omega = \mathbb{R}^N$ , or  $\Omega$  is a smooth bounded domain of  $\mathbb{R}^N$  and  $u = 0$  on  $\partial\Omega \times (0, T)$ .

In case  $\Omega = \mathbb{R}^N$ , we show that any solution  $u \in C^{2,1}(Q_{\mathbb{R}^N,T})$  of equation (1.1) in  $Q_{\mathbb{R}^N,T}$  (in particular any weak solution if  $q \leq 2$ ), without condition as  $|x| \rightarrow \infty$ , satisfies the universal estimate

$$|\nabla u(., t)|^q \leq \frac{1}{q-1} \frac{u(., t)}{t}, \quad \text{in } Q_{\mathbb{R}^N,T}.$$

Moreover we prove that the growth of  $u$  is limited by  $C(t+t^{-1/(q-1)})(1+|x|^{q'})$ , where  $C$  depends on  $u$ .

We also give existence properties of solutions in  $Q_{\Omega,T}$ , for initial data locally integrable or unbounded measures. We give a nonuniqueness result in case  $q > 2$ . Finally we show that besides the local regularizing effect of the heat equation,  $u$  satisfies a second effect of type  $L_{loc}^R - L_{loc}^\infty$ , due to the gradient term.

**Keywords** Hamilton-Jacobi equation; Radon measures; initial trace; universal bounds., regularizing effects.

**A.M.S. Subject Classification** 35K15, 35K55, 35B33, 35B65, 35D30

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# 1 Introduction

Here we consider the *nonnegative* solutions of the parabolic Hamilton-Jacobi equation

$$u_t - \nu \Delta u + |\nabla u|^q = 0, \quad (1.1)$$

where  $q > 1$ , in  $Q_{\Omega,T} = \Omega \times (0, T)$ , where  $\Omega$  is any domain of  $\mathbb{R}^N$ ,  $\nu \in (0, 1]$ . We study the problem of apriori estimates of the *nonnegative* solutions, with possibly rough *unbounded* initial data

$$u(x, 0) = u_0 \in \mathcal{M}^+(\Omega), \quad (1.2)$$

where we denote by  $\mathcal{M}^+(\Omega)$  the set of nonnegative Radon measures in  $\Omega$ , and  $\mathcal{M}_b^+(\Omega)$  the subset of bounded ones. We say that  $u$  is a solution of (1.1) if it satisfies (1.1) in  $Q_{\Omega,T}$  in the weak sense of distributions, see Section 2. We say that  $u$  has a trace  $u_0$  in  $\mathcal{M}^+(\Omega)$  if  $u(\cdot, t)$  converges to  $u_0$  in the weak\* topology of measures:

$$\lim_{t \rightarrow 0} \int_{\Omega} u(\cdot, t) \psi dx = \int_{\Omega} \psi du_0, \quad \forall \psi \in C_c(\Omega). \quad (1.3)$$

Our purpose is to obtain apriori estimates valid for any solution in  $Q_{\Omega,T} = \Omega \times (0, T)$ , without assumption on the boundary of  $\Omega$ , or for large  $|x|$  if  $\Omega = \mathbb{R}^N$ .

Fisrt recall some known results. The Cauchy problem in  $Q_{\mathbb{R}^N,T}$

$$(P_{\mathbb{R}^N,T}) \begin{cases} u_t - \nu \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\mathbb{R}^N,T}, \\ u(x, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.4)$$

is the object of a rich literature, see among them [2],[9], [5], [11], [26],[12], [13], and references therein. The first studies concern *classical* solutions, that means  $u \in C^{2,1}(Q_{\mathbb{R}^N,T})$ , with *smooth bounded initial data*  $u_0 \in C_b^2(\mathbb{R}^N)$ : there a unique global solution such that

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}, \text{ and } \|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \|\nabla u_0\|_{L^\infty(\mathbb{R}^N)}, \quad \text{in } Q_{\mathbb{R}^N,T},$$

see [2]. Then universal apriori estimates of the gradient are obtained *for this solution*, by using the Bersnstein technique, which consists in computing the equation satisfied by  $|\nabla u|^2$ : first from [23],

$$\|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}^q \leq \frac{\|u_0\|_{L^\infty(\mathbb{R}^N)}}{t},$$

in  $Q_{\mathbb{R}^N,T}$ , then from [9],

$$|\nabla u(\cdot, t)|^q \leq \frac{1}{q-1} \frac{u(\cdot, t)}{t}, \quad (1.5)$$

$$\|\nabla(u^{\frac{q-1}{q}})(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C t^{-1/2} \|u_0\|_{L^\infty(\mathbb{R}^N)}^{\frac{q-1}{q}}, \quad C = C(N, q, \nu). \quad (1.6)$$

Existence and uniqueness was extended to any  $u_0 \in C_b(\mathbb{R}^N)$  in [20]; then the estimates (1.6) and (1.5) are still valid, see [5]. In case of nonnegative rough initial data  $u_0 \in L^R(\mathbb{R}^N)$ ,  $R \geq 1$ , or  $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$ , the problem was studied in a semi-group formulation [9], [11], [26], then in the

larger class of weak solutions in [12], [13]. Recall that two critical values appear:  $q = 2$ , where the equation can be reduced to the heat equation, and

$$q_* = \frac{N+2}{N+1}.$$

Indeed the Cauchy problem with initial value  $u_0 = \kappa \delta_0$ , where  $\delta_0$  is the Dirac mass at 0 and  $\kappa > 0$ , has a weak solution  $u^\kappa$  if and only if  $q < q_*$ , see [9], [12]. Moreover as  $\kappa \rightarrow \infty$ ,  $(u^\kappa)$  converges to a unique very singular solution  $Y$ , see [25], [10], [8], [12]. And  $Y(x, t) = t^{-a/2} F(|x|/\sqrt{t})$ , where

$$a = \frac{2-q}{q-1}, \quad (1.7)$$

and  $F$  is bounded and has an exponential decay at infinity.

In [13, Theorem 2.2] it is shown that for any  $R \geq 1$  global regularizing  $L^R$ - $L^\infty$  properties of two types hold for the Cauchy problem in  $Q_{\mathbb{R}^N, T}$ : one due to the heat operator:

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C t^{-\frac{N}{2R}} \|u_0\|_{L^R(\mathbb{R}^N)}, \quad C = C(N, R, \nu), \quad (1.8)$$

and the other due to the gradient term, independent of  $\nu$  ( $\nu > 0$ ):

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C t^{-\frac{N}{qR+N(q-1)}} \|u_0\|_{L^R(\mathbb{R}^N)}^{\frac{qR}{qR+N(q-1)}}, \quad C = C(N, q, R). \quad (1.9)$$

A great part of the results has been extended to the Dirichlet problem in a bounded domain  $\Omega$ :

$$(P_{\Omega, T}) \begin{cases} u_t - \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega, T}, \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0, \end{cases} \quad (1.10)$$

where  $u_0 \in \mathcal{M}_b^+(\Omega)$ , and  $u(\cdot, t)$  converges to  $u_0$  weakly in  $\mathcal{M}_b^+(\Omega)$ , see [6], [26], [12], [13]. Universal estimates are given in [16], see also [12]. Note that (1.5) cannot hold, since it contradicts the Höpf Lemma.

Finally local estimates in any domain  $\Omega$  were proved in [26]: for any classical solution  $u$  in  $Q_{\Omega, T}$  and any ball  $B(x_0, 2\eta) \subset \Omega$ , there holds in  $Q_{B(x_0, \eta), T}$

$$|\nabla u|(\cdot, t) \leq C(t^{-\frac{1}{q}} + \eta^{-1} + \eta^{-\frac{1}{q-1}})(1 + u(\cdot, t)), \quad C = C(N, q, \nu). \quad (1.11)$$

## 1.1 Main results

In Section 3 we give *local integral estimates* of the solutions *in terms of the initial data, and a first regularizing effect*, local version of (1.8), see Theorem 3.3.

**Theorem 1.1** *Let  $q > 1$ . Let  $u$  be any nonnegative weak solution of equation (1.1) in  $Q_{\Omega, T}$ , and let  $B(x_0, 2\eta) \subset \subset \Omega$  such that  $u$  has a trace  $u_0 \in L_{loc}^R(\Omega)$ ,  $R \geq 1$  and  $u \in C([0, T]; L_{loc}^R(\Omega))$ . Then for any  $0 < t \leq \tau < T$ ,*

$$\sup_{x \in B(x_0, \eta/2)} u(x, t) \leq C t^{-\frac{N}{2R}} (t + \|u_0\|_{L^R(B(x_0, \eta))}), \quad C = C(N, q, \nu, R, \eta, \tau).$$

*If  $R = 1$ , the estimate remains true when  $u_0 \in \mathcal{M}^+(\Omega)$  (with  $\|u_0\|_{L^1(B(x_0, \eta))}$  replaced by  $\int_{B(x_0, \eta)} du_0$ ).*

In Section 4, we give *global estimates* of the solutions of (1.1) in  $Q_{\mathbb{R}^N, T}$ , and this is our main result. We show that *the universal estimate (1.5) in  $\mathbb{R}^N$  holds without assuming that the solutions are initially bounded*:

**Theorem 1.2** *Let  $q > 1$ . Let  $u$  be any classical solution, in particular **any weak solution** if  $q \leq 2$ , of equation (1.1) in  $Q_{\mathbb{R}^N, T}$ . Then*

$$|\nabla u(., t)|^q \leq \frac{1}{q-1} \frac{u(., t)}{t}, \quad \text{in } Q_{\mathbb{R}^N, T}. \quad (1.12)$$

And we prove that *the growth of the solutions is limited, in  $|x|^{q'}$  as  $|x| \rightarrow \infty$  and in  $t^{-1/(q-1)}$  as  $t \rightarrow 0$* :

**Theorem 1.3** *Let  $q > 1$ . Let  $u$  be any classical solution, in particular **any weak solution** if  $q \leq 2$ , of equation (1.1) in  $Q_{\mathbb{R}^N, T}$ , such that there exists a ball  $B(x_0, 2\eta)$  such that  $u$  has a trace  $u_0 \in \mathcal{M}^+(B(x_0, 2\eta))$ . Then*

$$u(x, t) \leq C(q) t^{-\frac{1}{q-1}} |x - x_0|^{q'} + C(t^{-\frac{1}{q-1}} + t + \int_{B(x_0, \eta)} du_0), \quad C = C(N, q, \eta). \quad (1.13)$$

In [14], we show that there exist solutions with precisely this type of behaviour of order  $t^{-1/(q-1)} |x|^{q'}$  as  $|x| \rightarrow \infty$  or  $t \rightarrow 0$ . Moreover we prove that the condition on the trace is always satisfied for  $q < q_*$ .

In Section 5 we complete the study by giving *existence results* with only *local assumptions* on  $u_0$ , extending some results of [5] where  $u_0$  is continuous, and [11], [13], where the assumptions are global:

**Theorem 1.4** *Let  $\Omega = \mathbb{R}^N$  (resp.  $\Omega$  bounded).*

(i) *If  $1 < q < q_*$ , then for any  $u_0 \in \mathcal{M}^+(\mathbb{R}^N)$  (resp.  $\mathcal{M}^+(\Omega)$ ), there exists a weak solution  $u$  of equation (1.1) (resp. of  $(D_{\Omega, T})$ ) with trace  $u_0$ .*

(ii) *If  $q_* \leq q \leq 2$ , then existence still holds for any nonnegative  $u_0 \in L^1_{loc}(\mathbb{R}^N)$  (resp.  $L^1_{loc}(\Omega)$ ). And then  $u \in C([0, T]; L^1_{loc}(\mathbb{R}^N))$  (resp.  $u \in C([0, T]; L^1_{loc}(\Omega))$ .*

(iii) *If  $q > 2$ , existence holds for any nonnegative  $u_0 \in L^1_{loc}(\mathbb{R}^N)$  (resp.  $L^1_{loc}(\Omega)$ ) which is limit of a nondecreasing sequence of continuous functions.*

Moreover we give a result of *nonuniqueness* of weak solutions in case  $q > 2$ :

**Theorem 1.5** *Assume that  $q > 2$ ,  $N \geq 2$ . Then the Cauchy problem  $(P_{\mathbb{R}^N, \infty})$  with initial data*

$$\tilde{U}(x) = \tilde{C} |x|^{|a|} \in C(\mathbb{R}^N), \quad \tilde{C} = \frac{q-1}{q-2} \left( \frac{(N-1)q-N}{q-1} \right)^{\frac{1}{q-1}},$$

*admits at least two weak solutions: the stationary solution  $\tilde{U}$ , and a radial self-similar solution of the form*

$$U_{\tilde{C}}(x, t) = t^{|a|/2} f(|x|/\sqrt{t}), \quad (1.14)$$

*where  $f$  is increasing on  $[0, \infty)$ ,  $f(0) > 0$ , and  $\lim_{\eta \rightarrow \infty} \eta^{-|a|/2} f(\eta) = \tilde{C}$ .*

Finally we give in Section 6 a second type of regularizing effects giving a local version of (1.9).

**Theorem 1.6** *Let  $q > 1$ , and let  $u$  be any nonnegative classical solution (resp. any weak solution if  $q \leq 2$ ) of equation (1.1) in  $Q_{\Omega,T}$ , and let  $B(x_0, 2\eta) \subset \Omega$ . Assume that  $u_0 \in L^R_{loc}(\Omega)$  for some  $R \geq 1$ ,  $R > q - 1$ , and  $u \in C([0, T]; L^R_{loc}(\Omega))$ . Then for any  $\varepsilon > 0$ , and for any  $\tau \in (0, T)$ , then there exists  $C = C(N, q, R, \eta, \varepsilon, \tau)$  such that*

$$\sup_{B_{\eta/2}} u(., t) \leq C t^{-\frac{N}{qR+N(q-1)}} (t + \|u_0\|_{L^R(B_\eta)})^{\frac{Rq}{qR+N(q-1)}} + C t^{\frac{1-\varepsilon}{R+1-q}} \|u_0\|_{L^R(B_\eta)}^{\frac{R}{R+1-q}}. \quad (1.15)$$

If  $q < 2$ , the estimates for  $R = 1$  are also valid when  $u$  has a trace  $u_0 \in \mathcal{M}^+(\Omega)$ , with  $\|u_0\|_{L^1(B_\eta)}$  replaced by  $\int_{B_\eta} du_0$ .

In conclusion, note that a part of our results could be extended to more general quasilinear operators, for example to the case of equation involving the  $p$ -Laplace operator

$$u_t - \nu \Delta_p u + |\nabla u|^q = 0$$

with  $p > 1$ , following the results of [13], [4], [21], [19].

## 2 Classical and weak solutions

We set  $Q_{\Omega,s,\tau} = \Omega \times (s, \tau)$ , for any  $0 \leq s < \tau \leq \infty$ , thus  $Q_{\Omega,T} = Q_{\Omega,0,T}$ .

**Definition 2.1** *Let  $q > 1$  and  $\Omega$  be any domain of  $\mathbb{R}^N$ . We say that a nonnegative function  $u$  is a **classical** solution of (1.1) in  $Q_{\Omega,T}$  if  $u \in C^{2,1}(Q_{\Omega,T})$ . We say that  $u$  is a **weak solution** (resp. weak subsolution) of (1.1) in  $Q_{\Omega,T}$ , if  $u \in C((0, T); L^1_{loc}(Q_{\Omega,T})) \cap L^1_{loc}((0, T); W^{1,1}_{loc}(\Omega))$ ,  $|\nabla u|^q \in L^1_{loc}(Q_{\Omega,T})$  and  $u$  satisfies (1.1) in the distribution sense:*

$$\int_0^T \int_\Omega (-u\varphi_t - \nu u \Delta \varphi + |\nabla u|^q \varphi) = 0, \quad \forall \varphi \in \mathcal{D}(Q_{\Omega,T}), \quad (2.1)$$

(resp.

$$\int_0^T \int_\Omega (-u\varphi_t - \nu u \Delta \varphi + |\nabla u|^q \varphi) \leq 0, \quad \forall \varphi \in \mathcal{D}^+(Q_{\Omega,T}).) \quad (2.2)$$

And then for any  $0 < s < t < T$ , and any  $\varphi \in C^1((0, T), C^1_c(\Omega))$ ,

$$\int_\Omega (u\varphi)(., t) - \int_\Omega (u\varphi)(., s) + \int_s^t \int_\Omega (-u\varphi_t + \nu \nabla u \cdot \nabla \varphi + |\nabla u|^q \varphi) = 0 \quad (\text{resp. } \leq 0). \quad (2.3)$$

**Remark 2.2** *Any weak subsolution  $u$  is locally bounded in  $Q_{\Omega,T}$ . Indeed, since  $u$  is  $\nu$ -subcaloric, there holds for any ball  $B(x_0, \rho) \subset \subset \Omega$  and any  $\rho^2 \leq t < T$ ,*

$$\sup_{B(x_0, \frac{\rho}{2}) \times [t - \frac{\rho^2}{4}, t]} u \leq C(N, \nu) \rho^{-(N+2)} \int_{t - \frac{\rho^2}{2}}^t \int_{B(x_0, \rho)} u. \quad (2.4)$$

Any nonnegative function  $u \in L^1_{loc}(Q_{\Omega,T})$ , such that  $|\nabla u|^q \in L^1_{loc}(Q_{\Omega,T})$ , and  $u$  satisfies (2.1), is a weak solution and  $|\nabla u| \in L^2_{loc}(Q_{\Omega,T})$ ,  $u \in C((0, T); L^s_{loc}(Q_{\Omega,T}))$ ,  $\forall s \geq 1$ , see [12, Lemma 2.4].

Next we recall the regularity of the weak solutions of (1.1) for  $q \leq 2$ , see [12, Theorem 2.9], [13, Corollary 5.14]:

**Theorem 2.3** *Let  $1 < q \leq 2$ . Let  $\Omega$  be any domain in  $\mathbb{R}^N$ . Let  $u$  be any weak nonnegative solution of (1.1) in  $Q_{\Omega,T}$ . Then  $u \in C_{loc}^{2+\gamma,1+\gamma/2}(Q_{\Omega,T})$  for some  $\gamma \in (0,1)$ , and for any smooth domains  $\omega \subset \subset \omega' \subset \subset \Omega$ , and  $0 < s < \tau < T$ ,  $\|u\|_{C^{2+\gamma,1+\gamma/2}(Q_{\omega,s,\tau})}$  is bounded in terms of  $\|u\|_{L^\infty(Q_{\omega',s/2,\tau})}$ . Thus for any sequence  $(u_n)$  of nonnegative weak solutions of equation (1.1) in  $Q_{\Omega,T}$ , uniformly locally bounded, one can extract a subsequence converging in  $C_{loc}^{2,1}(Q_{\Omega,T})$  to a weak solution  $u$  of (1.1) in  $Q_{\Omega,T}$ .*

**Remark 2.4** *Let  $q > 1$ . From the estimates (1.11), for any sequence of classical nonnegative solutions  $(u_n)$  of (1.1) in  $Q_{\Omega,T}$ , uniformly bounded in  $L_{loc}^\infty(Q_{\Omega,T})$ , one can extract a subsequence converging in  $C_{loc}^{2,1}(Q_{\mathbb{R}^N,T})$  to a classical solution  $u$  of (1.1).*

**Remark 2.5** *Let us mention some results of concerning the trace, valid for any  $q > 1$ , see [12, Lemma 2.14]. Let  $u$  be any nonnegative weak solution  $u$  of (1.1) in  $Q_{\Omega,T}$ . Then  $u$  has a trace  $u_0$  in  $\mathcal{M}^+(\Omega)$  if and only if  $u \in L_{loc}^\infty([0,T]; L_{loc}^1(\Omega))$ , and if and only if  $|\nabla u|^q \in L_{loc}^1(\Omega \times [0,T])$ . And then for any  $t \in (0,T)$ , and any  $\varphi \in C_c^1(\Omega \times [0,T])$ , and any  $\zeta \in C_c^1(\Omega)$ ,*

$$\int_{\Omega} u(\cdot, t) \varphi dx + \int_0^t \int_{\Omega} (-u \varphi_t + \nu \nabla u \cdot \nabla \varphi + |\nabla u|^q \varphi) = \int_{\Omega} \varphi(\cdot, 0) du_0, \quad (2.5)$$

$$\int_{\Omega} u(\cdot, t) \zeta + \int_0^t \int_{\Omega} (\nu \nabla u \cdot \nabla \zeta + |\nabla u|^q \zeta) = \int_{\Omega} \zeta du_0. \quad (2.6)$$

If  $u_0 \in L_{loc}^1(\Omega)$ , then  $u \in C([0,T]; L_{loc}^1(\Omega))$ .

Finally we consider the Dirichlet problem in a smooth bounded domain  $\Omega$ :

$$(D_{\Omega,T}) \begin{cases} u_t - \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega,T}, \\ u = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \quad (2.7)$$

**Definition 2.6** *We say that a function  $u$  is a **weak solution** of  $(D_{\Omega,T})$  if it is a weak solution of equation (1.1) such that  $u \in C((0,T); L^1(\Omega)) \cap L_{loc}^1((0,T); W_0^{1,1}(\Omega))$ , and  $|\nabla u|^q \in L_{loc}^1((0,T); L^1(\Omega))$ . We say that  $u$  is a **classical solution** of  $(D_{\Omega,T})$  if  $u \in C^{2,1}(Q_{\Omega,T}) \cap C^{1,0}(\overline{\Omega} \times (0,T))$ .*

### 3 Local integral properties and first regularizing effect

#### 3.1 Local integral properties

**Lemma 3.1** *Let  $\Omega$  be any domain in  $\mathbb{R}^N$ ,  $q > 1, R \geq 1$ . Let  $u$  be any nonnegative weak subsolution of equation (1.1) in  $Q_{\Omega,T}$ , such that  $u \in C((0,T); L_{loc}^R(\Omega))$ . Let  $\xi \in C^1((0,T); C_c^1(\Omega))$ , with values in  $[0,1]$ . Let  $\lambda > 1$ . Then there exists  $C = C(q, R, \lambda)$ , such that, for any  $0 < s < t \leq \tau < T$ ,*

$$\begin{aligned} & \int_{\Omega} u^R(\cdot, t) \xi^\lambda + \frac{1}{2} \int_s^\tau \int_{\Omega} u^{R-1} |\nabla u|^q \xi^\lambda + \nu \frac{R-1}{2} \int_s^\tau \int_{\Omega} u^{R-2} |\nabla u|^2 \xi^\lambda \\ & \leq \int_{\Omega} u^R(\cdot, s) \xi^\lambda + \lambda R \int_s^t \int_{\Omega} u^R \xi^{\lambda-1} |\xi_t| + C \int_s^t \int_{\Omega} u^{R-1} \xi^{\lambda-q'} |\nabla \xi|^{q'}. \end{aligned} \quad (3.1)$$

**Proof.** (i) Let  $R = 1$ . Taking  $\varphi = \xi^\lambda$  in (2.3), we obtain, since  $\nu \leq 1$ ,

$$\begin{aligned} \int_{\Omega} u(., t) \xi^\lambda + \int_s^t \int_{\Omega} |\nabla u|^q \xi^\lambda &\leq \int_{\Omega} u(s, .) \xi^\lambda + \lambda \int_s^t \int_{\Omega} \xi^{\lambda-1} u \xi_t + \lambda \nu \int_s^t \int_{\Omega} \xi^{\lambda-1} \nabla u \cdot \nabla \xi \\ &\leq \int_{\Omega} u(., s) \xi^\lambda + \lambda \int_s^t \int_{\Omega} \xi^{\lambda-1} u |\xi_t| + \frac{1}{2} \int_s^t \int_{\Omega} |\nabla u|^q \xi^{q'} + C(q, \lambda) \int_s^t \int_{\Omega} \xi^{\lambda-q'} |\nabla \xi|^{q'}, \end{aligned}$$

hence (3.1) follows.

(ii) Next assume  $R > 1$ . Consider  $u_{\delta, n} = ((u + \delta) * \varphi_n)$ , where  $(\varphi_n)$  is a sequence of mollifiers, and  $\delta > 0$ . Then by convexity,  $u_{\delta, n}$  is also a subsolution of (1.1):

$$(u_{\delta, n})_t - \nu \Delta u_{\delta, n} + |\nabla u_{\delta, n}|^q \leq 0.$$

Multiplying by  $u_{\delta, n}^{R-1} \xi^\lambda$  and integrating between  $s$  and  $t$ , and going to the limit as  $\delta \rightarrow 0$  and  $n \rightarrow \infty$ , see [13], we get with different constants  $C = (N, q, R, \lambda)$ , independent of  $\nu$ ,

$$\begin{aligned} &\frac{1}{R} \int_{\Omega} u^R(., t) \xi^\lambda + \nu(R-1) \int_s^t \int_{\Omega} u^{R-2} |\nabla u|^2 \xi^\lambda + \int_s^t \int_{\Omega} u^{R-1} |\nabla u|^q \xi^\lambda \\ &\leq \frac{1}{R} \int_{\Omega} u^R(., s) \xi^\lambda + \lambda \int_s^t \int_{B_\rho} \xi^{\lambda-1} u^R |\xi_t| + \lambda \nu \int_s^t \int_{\Omega} u^{R-1} |\nabla u| |\nabla \xi| \xi^{\lambda-1} \\ &\leq \frac{1}{R} \int_{\Omega} u^R(., s) \xi^\lambda + \lambda \int_s^t \int_{B_\rho} \xi^{\lambda-1} u^R |\xi_t| \\ &\quad + \frac{1}{2} \int_s^t \int_{\Omega} u^{R-1} |\nabla u|^q \xi^\lambda + C(\lambda, R) \int_s^t \int_{\Omega} u^{R-1} \xi^{\lambda-q'} |\nabla \xi|^{q'}, \end{aligned}$$

and (3.1) follows again. ■

Then we give local integral estimates of  $u(., t)$  in terms of the initial data:

**Lemma 3.2** *Let  $q > 1$ . Let  $\eta > 0$ . Let  $u$  be any nonnegative weak solution of equation (1.1) in  $Q_{\Omega, T}$ , with trace  $u_0 \in \mathcal{M}^+(\Omega)$ , and let  $B(x_0, 2\eta) \subset\subset \Omega$ . Then for any  $t \in (0, T)$ ,*

$$\int_{B(x_0, \eta)} u(x, t) \leq C(N, q) \eta^{N-q'} t + \int_{B(x_0, 2\eta)} du_0. \quad (3.2)$$

Moreover if  $u_0 \in L_{loc}^R(\Omega)$  ( $R > 1$ ), and  $u \in C([0, T]; L_{loc}^R(\Omega))$ , then

$$\|u(., t)\|_{L^R(B(x_0, \eta))} \leq C(N, q, R) \eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B(x_0, 2\eta))}. \quad (3.3)$$

If  $u \in C(\overline{B(x_0, 2\eta)} \times [0, T])$ , then

$$\|u(., t)\|_{L^\infty(B(x_0, \eta))} \leq C(N, q) \eta^{-q'} t + \|u_0\|_{L^\infty(B(x_0, 2\eta))}. \quad (3.4)$$

**Proof.** We can assume that  $0 \in \Omega$  and  $x_0 = 0$ . We take  $\xi \in C_c^1(\Omega)$ , independent of  $t$ , with values in  $[0, 1]$ , and  $R = 1$  in (3.1),  $\lambda = q'$ . Then for any  $0 < s < t < T$ ,

$$\int_{\Omega} u(., t) \xi^{q'} + \frac{1}{2} \int_s^t \int_{\Omega} |\nabla u|^q \xi^{q'} \leq \int_{\Omega} u(., s) \xi^{q'} + C(q) \int_s^t \int_{\Omega} |\nabla \xi|^{q'} \leq \int_{\Omega} u(., s) \xi^{q'} + C(q) t \int_{\Omega} |\nabla \xi|^{q'}.$$



Hence as  $s \rightarrow 0$ , we get

$$\int_{\Omega} u(., t) \xi^{q'} + \frac{1}{2} \int_0^t \int_{\Omega} |\nabla u|^q \xi^{q'} \leq C(q)t \int_{\Omega} |\nabla \xi|^{q'} + \int_{\Omega} \xi^{q'} du_0. \quad (3.5)$$

Then taking  $\xi = 1$  in  $B_{\eta}$  with support in  $B_{2\eta}$  and  $|\nabla \xi| \leq C_0(N)/\eta$ ,

$$\int_{B_{\eta}} u(x, t) \leq C(N, q) \eta^{N-q'} t + \int_{B_{2\eta}} \xi^{q'} du_0, \quad (3.6)$$

hence we get (3.2). Next assume  $u_0 \in L_{loc}^R(\Omega)$  ( $R > 1$ ), and  $u \in C([0, T]; L_{loc}^R(\Omega))$ . Then from (3.1), for any  $0 < s < t \leq \tau < T$ , we find,

$$\begin{aligned} \int_{\Omega} u^R(., t) \xi^{\lambda} + \frac{1}{2} \int_s^{\tau} \int_{\Omega} u^{R-1} |\nabla u|^q \xi^{\lambda} &\leq \int_{\Omega} u^R(., s) \xi^{\lambda} + \int_s^t \int_{\Omega} u^{R-1} \xi^{\lambda-q'} |\nabla \xi|^{q'} \\ &\leq \int_{\Omega} u^R(., s) \xi^{\lambda} + \varepsilon \int_s^t \int_{B_{2\eta}} u^R \xi^{\lambda} + \varepsilon^{1-R} \int_s^t \int_{B_{2\eta}} \xi^{\lambda-Rq'} |\nabla \xi|^{Rq'}. \end{aligned}$$

Taking  $\lambda = Rq'$ , and  $\xi$  as above, we find

$$\int_{B_{2\eta}} u^R(., t) \xi^{Rq'} \leq \int_{B_{2\eta}} u^R(., s) \xi^{Rq'} + \varepsilon \int_s^t \int_{B_{2\eta}} u^R \xi^{Rq'} + \varepsilon^{1-R} C(N) C_0^{Rq'}(N) \eta^{N-Rq'} t.$$

Next we set  $\varpi(t) = \sup_{\sigma \in [s, t]} \int_{B_{2\eta}} u^R(., \sigma) \xi^{Rq'}$ . Then

$$\varpi(t) \leq \int_{B_{2\eta}} u^R(., s) \xi^{Rq'} + \varepsilon(t-s) \varpi(t) + \varepsilon^{1-R} C(N) C_0^{Rq'}(N) \eta^{N-Rq'} t.$$

Taking  $\varepsilon = 1/2t$ , we get

$$\frac{1}{2} \int_{B_{2\eta}} u^R(., t) \xi^{Rq'} \leq \int_{B_{2\eta}} u^R(., s) \xi^{Rq'} + C(N) C_0^{Rq'}(N) \eta^{N-Rq'} t^R.$$

Then going to the limit as  $s \rightarrow 0$ ,

$$\int_{B_{\eta}} u^R(x, t) \leq C(N) C_0^{Rq'}(N) \eta^{N-Rq'} t^R + \int_{B_{2\eta}} u_0^R \xi^{Rq'}, \quad (3.7)$$

thus (3.3) follows.

If  $u \in C(\overline{B_{2\rho}} \times [0, T])$ , then (3.7) holds for any  $R \geq 1$ , implying

$$\|u(., t)\|_{L^R(B_{\eta})} \leq C^{\frac{1}{R}}(N) C_0^{q'}(N) \eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_{2\eta})},$$

and (3.3) follows as  $R \rightarrow \infty$ . ■

### 3.2 Regularizing effect of the heat operator

We first give a first regularizing effect due to the Laplace operator in  $Q_{\Omega,T}$ , for any domain  $\Omega$ , for classical or weak solutions in terms of the initial data.

**Theorem 3.3** *Let  $q > 1$ . Let  $u$  be any nonnegative weak subsolution of equation (1.1) in  $Q_{\Omega,T}$ , and let  $B(x_0, 2\eta) \subset \Omega$  such that  $u$  has a trace  $u_0 \in \mathcal{M}^+(B(x_0, 2\eta))$ . Then for any  $\tau < T$ , and any  $t \in (0, \tau]$ ,*

$$\sup_{x \in B(x_0, \eta/2)} u(x, t) \leq Ct^{-\frac{N}{2}} (t + \int_{B(x_0, \eta)} du_0), \quad C = C(N, q, \nu, \eta, \tau). \quad (3.8)$$

Moreover if  $u_0 \in L_{loc}^R(\Omega)$  ( $R > 1$ ), and  $u \in C([0, T]; L_{loc}^R(\Omega))$ , then

$$\sup_{x \in B(x_0, \eta/2)} u(x, t) \leq Ct^{-\frac{N}{2R}} (t + \|u_0\|_{L^R(B(x_0, \eta))}), \quad C = C(N, q, \nu, R, \eta, \tau). \quad (3.9)$$

**Proof.** We still assume that  $x_0 = 0 \in \Omega$ . Let  $\xi \in C_c^1(B_{2\eta})$  be nonnegative, radial, with values in  $[0, 1]$ , with  $\xi = 1$  on  $B_\eta$  and  $|\nabla \xi| \leq C_0(N)/\eta$ . Since  $u$  is  $\nu$ -subcaloric, from (2.4), for any  $\rho \in (0, \eta)$  such that  $\rho^2 \leq t < \tau$ ,

$$\sup_{B_{\eta/2}} u(., t) \leq C(N, \nu) \rho^{-(N+2)} \int_{t-\rho^2/4}^t \int_{B_\eta} u, \quad (3.10)$$

hence from Lemma 3.2,

$$\sup_{B_{\eta/2}} u(., t) \leq C(N, q, \nu) \rho^{-N} (\eta^{N-q'} t + \int_{B_{2\eta}} du_0).$$

Let  $k_0 \in \mathbb{N}$  such that  $k_0 \eta^2/2 \geq \tau$ . For any  $t \in (0, \tau]$ , there exists  $k \in \mathbb{N}$  with  $k \leq k_0$  such that  $t \in (k\eta^2/2, (k+1)\eta^2/2]$ . Taking  $\rho^2 = t/(k+1)$ , we find

$$\begin{aligned} \sup_{B_{\eta/2}} u(., t) &\leq C(N, q, \nu) (k_0 + 1)^{\frac{N}{2}} t^{-\frac{N}{2}} (\eta^{N-q'} t + \int_{B_{2\eta}} du_0) \\ &\leq C(N, q, \nu) (\eta^{-N} \tau^{\frac{N}{2}} + 1) t^{-\frac{N}{2}} (\eta^{N-q'} t + \int_{B_{2\eta}} du_0). \end{aligned} \quad (3.11)$$

Thus we obtain (3.8). Next assume that  $u \in C([0, T]; L_{loc}^R(B_{2\eta}))$ , with  $R > 1$ . We still approximate  $u$  by  $u_{\delta, n} = (u + \delta) * \varphi_n$ , where  $(\varphi_n)$  is a sequence of mollifiers, and  $\delta > 0$ . Since  $u$  is  $\nu$ -subcaloric, then  $u_{\delta, n}^R$  is also  $\nu$ -subcaloric. Then for any  $\rho \in (0, \eta)$  such that  $\rho^2 \leq t < \tau$ , we have

$$\sup_{B_{\eta/2}} u_{\delta, n}^R(., t) \leq C(N, \nu) \rho^{-(N+2)} \int_{t-\rho^2/4}^t \int_{B_{\rho/2}} u_{\delta, n}^R,$$

hence as  $\delta \rightarrow 0$  and  $n \rightarrow \infty$ , from Lemma (3.2),

$$\sup_{B_{\eta/2}} u^R(., t) \leq C(N, \nu) \rho^{-(N+2)} \int_{t-\rho^2/4}^t \int_{B_{\rho/2}} u^R \leq C(N, q, \nu, R) (\eta^{-N} \tau^{\frac{N}{2}} + 1) (\eta^{N-Rq'} t^R + \int_{B_{2\eta}} u_0^R). \quad (3.12)$$

We deduce (3.9) as above.  $\blacksquare$

## 4 Global estimates in $\mathbb{R}^N$

We first show that the universal estimate of the gradient (1.12) implies the estimate (1.13) of the function:

**Theorem 4.1** *Let  $q > 1$ . Let  $u$  be a classical solution of equation (1.1) in  $Q_{\mathbb{R}^N, T}$ . Assume that there exists a ball  $B(x_0, 2\eta)$  such that  $u$  has a trace  $u_0 \in \mathcal{M}^+(B(x_0, 2\eta))$ . If  $u$  satisfies (1.12), then for any  $t \in (0, T)$ ,*

$$u(x, t) \leq C(q)t^{-\frac{1}{q-1}} |x - x_0|^{q'} + C(t^{-\frac{1}{q-1}} + t + \int_{B(x_0, \eta)} du_0), \quad C = C(N, q, \eta), \quad (4.1)$$

If  $u_0 \in L_{loc}^R(\Omega)$ ,  $R \geq 1$  and  $u \in C([0, T]; L_{loc}^R(\Omega))$ , then

$$u(x, t) \leq C(q)t^{-\frac{1}{q-1}} |x - x_0|^{q'} + Ct^{-\frac{N}{2R}}(t + \|u_0\|_{L^R(B(x_0, \eta))}), \quad C = C(N, q, R, \nu, \eta). \quad (4.2)$$

$$u(x, t) \leq C(q)t^{-\frac{1}{q-1}} |x - x_0|^{q'} + C(t^{-\frac{1}{q-1}} + t + \|u_0\|_{L^R(B(x_0, \eta))}), \quad C = C(N, q, R, \eta). \quad (4.3)$$

**Proof.** Estimate (1.12) is equivalent to

$$\left| \nabla(u^{\frac{1}{q}}) \right|(\cdot, t) \leq \frac{(q-1)^{\frac{1}{q}}}{q} t^{-\frac{1}{q}}, \quad \text{in } Q_{\mathbb{R}^N, T}. \quad (4.4)$$

Then with constants  $C(q)$  only depending of  $q$ ,

$$u^{\frac{1}{q}}(x, t) \leq u^{\frac{1}{q}}(x_0, t) + C(q)t^{-\frac{1}{q}} |x - x_0|, \quad (4.5)$$

then

$$u(x, t) \leq C(q)(u(x_0, t) + t^{-\frac{1}{q-1}} |x - x_0|^{q'}), \quad (4.6)$$

and, from Theorem 3.3,

$$u(x_0, t) \leq C(N, q, R, \nu, \eta)t^{-\frac{N}{2R}}(t + \|u_0\|_{L^R(B(x_0, \eta))}).$$

Therefore (4.2) follows. Also, interverting  $x$  and  $x_0$ , for any  $R \geq 1$ ,

$$u^R(x_0, t) \leq C(q, R)(u^R(x, t) + t^{-\frac{R}{q-1}} |x - x_0|^{Rq'}).$$

Integrating on  $B(x_0, \eta/2)$ , we get

$$\eta^N u^R(x_0, t) \leq C(q, R) \left( \int_{B(x_0, \eta/2)} u^R(\cdot, t) + t^{-\frac{R}{q-1}} \eta^{N-Rq'} \right);$$

using Lemma 3.2, we deduce

$$u(x_0, t) \leq C(N, q, R, \eta)(t^{-\frac{1}{q-1}} + t + \int_{B(x_0, \eta)} du_0),$$

and if  $u_0 \in L_{loc}^R(\Omega)$ ,

$$u(x_0, t) \leq C(N, q, R, \eta)(t^{-\frac{1}{q-1}} + t + \|u_0\|_{L^R(B(x_0, \eta))}),$$

and the conclusions follow from (4.6). ■

**Remark 4.2** In particular, the estimates (4.1)-(4.3) hold for solutions with  $u_0 \in C_b(\mathbb{R}^N)$ , and more generally for limits a.e. of such solutions, that we can call **reachable** solutions. Inequality (4.5) was used in [5, Theorem 3.3] for obtaining local estimates of classical of bounded solutions in  $Q_{\mathbb{R}^N, T}$ .

In order to prove Theorem 1.2, we first give an estimate of the type of (1.13) on a time interval  $(0, \tau]$ , with constants depending on  $\tau$  and  $\nu$ , which is not obtained from any estimate of the gradient. Our result is based on the construction of suitable supersolutions in annulus of type  $Q_{B_{3\rho} \setminus \overline{B_\rho}, \infty}$ ,  $\rho > 0$ . For the construction we consider the function  $t \in (0, \infty) \mapsto \psi_h(t) \in (1, \infty)$ , where  $h > 0$  is a parameter, solution of the problem

$$(\psi_h)_t + h(\psi_h^q - \psi_h) = 0 \quad \text{in } (0, \infty), \quad \psi_h(0) = \infty, \quad \psi_h(\infty) = 1, \quad (4.7)$$

given explicitly by  $\psi_h(t) = (1 - e^{-h(q-1)t})^{-\frac{1}{q-1}}$ ; hence  $\psi_h^q - \psi_h \geq 0$ , and for any  $t > 0$ ,

$$((q-1)ht)^{-\frac{1}{q-1}} \leq \psi_h(t) \leq 2^{\frac{1}{q-1}}(1 + ((q-1)ht)^{-\frac{1}{q-1}}). \quad (4.8)$$

since, for  $x > 0$ ,  $x(1 - x/2) \leq 1 - e^{-x} \leq x$ , hence  $x/2 \leq 1 - e^{-x} \leq x$ , for  $x \leq 1$ .

**Proposition 4.3** Let  $q > 1$ . Then there exists a nonnegative function  $V$  defined in  $Q_{B_3 \times (0, \infty)}$ , such that  $V$  is a supersolution of equation (1.1) on  $Q_{B_3 \setminus \overline{B_1}, \infty}$ , and  $V$  converges to  $\infty$  as  $t \rightarrow 0$ , uniformly on  $B_3$  and converges to  $\infty$  as  $x \rightarrow \partial B_3$ , uniformly on  $(0, \tau)$  for any  $\tau < \infty$ . And  $V$  has the form

$$V(x, t) = e^t \Phi(|x|) \psi_h(t) \quad \text{in } Q_{B_3, \infty} \quad (4.9)$$

for some  $h = h(N, q, \nu) > 0$ , where  $\psi_h$  is given by (4.7), and  $\Phi$  is a suitable radial function depending on  $N, q, \nu$ , such that

$$-\nu \Delta \Phi + \Phi + |\nabla \Phi|^q \geq 0 \quad \text{in } B_3. \quad (4.10)$$

**Proof.** We first construct  $\Phi$ . Let  $\sigma > 0$ , such that  $\sigma \geq a = (2 - q)/(q - 1)$ . Let  $\varphi_1$  be the first eigenfunction of the Laplacian in  $B_3$  such that  $\varphi_1(0) = 1$ , associated to the first eigenvalue  $\lambda_1$ , hence  $\varphi_1$  is radial; let  $m_1 = \min_{\overline{B_1}} \varphi_1 > 0$  and  $M_1 = \min_{\overline{B_3} \setminus B_1} |\nabla \varphi_1|$ . Let us take  $\Phi = \Phi_K = \Phi_0 + K$ , where  $\Phi_0 = \gamma \varphi_1^{-\sigma}$ ,  $K > 0$  and  $\gamma > 0$  are parameters. Then

$$-\nu \Delta \Phi + \Phi + |\nabla \Phi|^q = F(\Phi_0) + K, \quad \text{with}$$

$$F(\Phi_0) = \gamma \varphi_1^{-(\sigma+2)} (\gamma^{q-1} \sigma^q \varphi_1^{(q-1)(a-\sigma)} |\varphi_1'|^q + (1 - \nu \sigma \lambda_1) \varphi_1^2 - \nu \sigma (\sigma + 1) \varphi_1'^2).$$

There holds  $\lim_{r \rightarrow 3} |\varphi_1'| = c_1 > 0$  from the Höpf Lemma. Taking  $\sigma > a$  we fix  $\gamma = 1$ , and then  $\lim_{r \rightarrow 3} F(\Phi_0) = \infty$ . If  $q < 2$  we can also take  $\sigma = a$ , we get

$$F(\Phi_0) = \gamma \varphi_1^{-q'} (\gamma^{q-1} a^q |\varphi_1'|^q + (1 - \nu a \lambda_1) \varphi_1^2 - a q' \varphi_1'^2),$$

hence fixing  $\gamma > \gamma(N, q, \nu)$  large enough, we still get  $\lim_{r \rightarrow 3} F(\Phi_0) = \infty$ . Thus  $F$  has a minimum  $\mu$  in  $B_3$ . Taking  $K = K(N, q, \nu) > |\mu|$  we deduce that  $\Phi$  satisfies (4.10), and  $\lim_{r \rightarrow 3} \Phi = \infty$ .

Observe that  $\Phi'^q/\Phi = \gamma^q \sigma^q / (\gamma \varphi_1^{q+\sigma(q-1)} + K \varphi_1^{q(\sigma+1)})$  is increasing, then  $m_K = m_K(N, q, \nu) = \min_{[1,3]} |\Phi'|^q / \Phi = |\Phi'(1)|^q / \Phi(1) > 0$ . We define  $V$  by (4.9) and compute

$$\begin{aligned} V_t - \nu \Delta V + |\nabla V|^q &= e^t (\Phi \psi_h + \Phi(\psi_h)_t - \nu \Delta \Phi) + e^{qt} |\nabla \Phi|^q \psi_h^q \\ &\geq e^t (\Phi \psi_h + \Phi \psi_t - \nu \Delta \Phi + |\nabla \Phi|^q \psi^q) = e^t (\psi^q - \psi_h) (|\nabla \Phi|^q - h \Phi). \end{aligned}$$

We take  $h = h(N, q, \nu) < m_K$ . Then on  $B_3 \setminus B_1$  we have  $|\nabla \Phi|^q - h \Phi > 0$ , and  $\psi^q \geq \psi_h$ , then  $V$  is a supersolution on  $B_3 \setminus B_1$ . Moreover  $V$  is radial and increasing with respect to  $|x|$ , then

$$\begin{aligned} \sup_{\overline{B_2}} V(x, t) &= \sup_{\partial B_2} V(x, t) = e^t \Phi(2) \psi_h(t) \leq 2^{\frac{1}{q-1}} e^t \Phi(2) (1 + ((q-1)ht)^{-\frac{1}{q-1}}) \\ &\leq C(N, q, \nu) e^t \Phi(2) (1 + t^{-\frac{1}{q-1}}). \end{aligned} \quad (4.11)$$

■

**Theorem 4.4** *Let  $u$  be a classical solution, (in particular any weak solution if  $q \leq 2$ ) of equation (1.1) in  $Q_{\mathbb{R}^N, T}$ . Assume that there exists a ball  $B(x_0, 2\eta)$  such that  $u$  admits a trace  $u_0 \in \mathcal{M}^+(B(x_0, 2\eta))$ .*

(i) *Then for any  $\tau \in (0, T)$ , and  $t \leq \tau$ ,*

$$u(x, t) \leq C(t^{-\frac{1}{q-1}} |x - x_0|^{q'} + t^{-\frac{N}{2}} (t + \int_{B(x_0, \eta)} du_0)), \quad C = C(N, q, \nu, \eta, \tau), \quad (4.12)$$

(ii) *Also if  $u \in C([0, T]; L_{loc}^R(B(x_0, 2\eta)))$ ,*

$$u(x, t) \leq C(t^{-\frac{1}{q-1}} |x - x_0|^{q'} + t^{-\frac{N}{2R}} (t + \|u_0\|_{L^R(B(x_0, \eta))})), \quad C = C(N, q, \nu, R, \eta, \tau), \quad (4.13)$$

*if  $u \in C([0, T] \times B(x_0, 2\eta))$ , then*

$$u(x, t) \leq C(t^{-\frac{1}{q-1}} |x - x_0|^{q'} + t + \sup_{B(x_0, \eta)} u_0), \quad C = C(N, q, \nu, \eta, \tau). \quad (4.14)$$

**Proof.** We use the function  $V$  constructed above. We can assume  $x_0 = 0$ . For any  $\rho > 0$ , we consider the function  $V_\rho$  defined in  $B_{3\rho} \times (0, \infty)$  by

$$V_\rho(x, t) = \rho^{-a} V(\rho^{-1}x, \rho^{-2}t).$$

It is a supersolution of the equation (1.1) on  $B_{3\rho} \setminus \overline{B_\rho} \times (0, \infty)$ , infinite on  $\partial B_{3\rho} \times (0, \infty)$  and on  $B_{3\rho} \times \{0\}$ , and from (4.11)

$$\begin{aligned} \sup_{\overline{B_{2\rho}}} V_\rho(x, t) &= \sup_{\partial B_{2\rho}} V_\rho(x, t) \leq C_1(N, q, \nu) \rho^{-a} e^{\frac{t}{\rho^2}} \Phi(2) (1 + \rho^{\frac{2}{q-1}} t^{-\frac{1}{q-1}}) \\ &\leq C_2(N, q, \nu) \rho^{q'} e^{\frac{t}{\rho^2}} (\rho^{-\frac{2}{q-1}} + t^{-\frac{1}{q-1}}). \end{aligned} \quad (4.15)$$

(i) First suppose that  $u \in C([0, T] \times \mathbb{R}^N)$ . Let  $\tau \in (0, T)$ , and  $C(\tau) = \sup_{Q_{B_\rho, \tau}} u$ . Then  $w = C(\tau) + V_\rho$  is a supersolution in  $Q = (B_{3\rho} \setminus \overline{B_\rho}) \times (0, \tau]$ , and from the comparison principle we obtain  $u \leq C(\tau) + V_\rho$  in that set. Indeed let  $\epsilon > 0$  small enough. Then there exists  $\tau_\epsilon < \epsilon$  and

$r_\epsilon \in (3\rho - \epsilon, 3\rho)$ , such that  $w(., s) \geq \max_{\overline{B_{3\rho}}} u(., \epsilon)$  for any  $s \in (0, \tau_\epsilon]$ , and  $w(x, t) \geq \max_{\overline{B_{3\rho}} \times [0, \tau]} u$  for any  $t \in (0, \tau]$  and  $r_\epsilon \leq |x| < 3\rho$ . We compare  $u(x, t + \epsilon)$  with  $w(x, t + s)$  on  $[0, \tau - \epsilon] \times \overline{B_{r_\epsilon}} \setminus \overline{B_\rho}$ . And for  $|x| = \rho$ , we have  $u(x, t + \epsilon) \leq C(\tau) \leq w(x, t + s)$ . Then  $u(., t + \epsilon) \leq w(., t + s)$  in  $\overline{B_{r_\epsilon}} \setminus \overline{B_\rho} \times (0, \tau - \epsilon]$ . As  $s, \epsilon \rightarrow 0$ , we deduce that  $u \leq w$  in  $Q$ .

Hence in  $\overline{B_{2\rho}} \times (0, \tau)$ , we find from (4.15)

$$u \leq C(\tau) + \sup_{\overline{B_{2\rho}}} V_\rho(x, t) \leq C(\tau) + C_2 \rho^{q'} e^{\frac{t}{\rho^2}} (\rho^{-\frac{2}{q-1}} + t^{-\frac{1}{q-1}}). \quad (4.16)$$

Making  $t$  tend to  $\tau$ , this proves that

$$\sup_{Q_{B_{2\rho}, \tau}} u \leq \sup_{Q_{B_\rho, \tau}} u + C_2 \rho^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}})$$

By induction, we get

$$\begin{aligned} \sup_{Q_{B_{2^{n+1}\rho}, \tau}} u &\leq \sup_{Q_{B_{2^n\rho}, \tau}} u + C_2 2^{nq'} \rho^{q'} e^{\frac{\tau}{4^n \rho^2}} ((2^n \rho)^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}) \\ &\leq \sup_{Q_{B_{2^n\rho}, \tau}} u + C_2 2^{nq'} \rho^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}); \end{aligned}$$

$$\begin{aligned} \sup_{Q_{B_{2^{n+1}\rho}, \tau}} u &\leq \sup_{Q_{B_\rho}} u + C_2 (1 + 2^{q'} + \dots + 2^{nq'}) \rho^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}) \\ &\leq \sup_{Q_{B_\rho, \tau}} u + C_2 2^{q'} (2^n \rho)^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}). \end{aligned}$$

For any  $x \in \mathbb{R}^N$  such that  $|x| \geq \rho$ , there exists  $n \in \mathbb{N}^*$  such that  $x \in B_{2^{n+1}\rho} \setminus \overline{B_{2^n\rho}}$ , then

$$u(x, \tau) \leq \sup_{Q_{B_\rho, \tau}} u + C_2 2^{q'} |x|^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}) \quad (4.17)$$

thus

$$\sup_{Q_{\mathbb{R}^N, \tau}} u \leq \sup_{Q_{B_\rho, \tau}} u + C_2 2^{q'} |x|^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}). \quad (4.18)$$

(ii) Next we consider any classical solution  $u$  in  $Q_{\mathbb{R}^N, T}$  with trace  $u_0$  in  $B(x_0, 2\eta)$ . We still assume  $x_0 = 0$ . Then for  $0 < \epsilon \leq t \leq \tau$ , from (3.4) in Lemma 3.2, there holds

$$\sup_{B_{\eta/2}} u(x, t) \leq C(N, q) \eta^{-q'} t + \sup_{B_\eta} u(x, \epsilon).$$

Then from (4.18) with  $\rho = \eta/2$ , we deduce that for any  $(x, t) \in Q_{\mathbb{R}^N, \epsilon, \tau}$ ,

$$u(x, t) \leq C(N, q) \eta^{-q'} t + \sup_{B_{\eta/2}} u(., \epsilon) + C(1 + (t - \epsilon)^{-\frac{1}{q-1}}) |x|^{q'},$$

with  $C = C(N, q, \nu, \eta, \tau)$ . Next we take  $\epsilon = t/2$ . Then for any  $t \in (0, \tau]$ , from (3.8) in Theorem 3.3,

$$u(x, t) \leq C(N, q, \eta)t + Ct^{-1(q-1)}|x|^{q'} + Ct^{-\frac{N}{2}}(t + \int_{B_\eta} du_0).$$

with  $C = C(N, q, \nu, \eta, \tau)$  and we obtain (4.12). And (4.13), (4.14) follow from (3.9) and (3.4).  $\blacksquare$

Next we show our main Theorem 1.2. We use a *local* Bernstein technique, as in [26]. The idea is to compute the equation satisfied by the function  $v = u^{(q-1)/q}$ , introduced in [9], and the equation satisfied by  $w = |\nabla v|^2$ , to obtain estimates of  $w$  in a cylinder  $Q_{B_M, T}$ ,  $M > 0$ . The difficulty is that this equation involves an elliptic operator  $w \mapsto w_t - \Delta w + b \cdot \nabla w$ , where  $b$  depends on  $v$ , and may be unbounded. However it can be controlled by the estimates of  $v$  obtained at Theorem 4.4. Then as  $M \rightarrow \infty$ , we can prove nonuniversal  $L^\infty$  estimates of  $w$ . Finally we obtain universal estimates of  $w$  by application of the maximum principle in  $Q_{\mathbb{R}^N, T}$ , valid because  $w$  is bounded. First we give a slight improvement of a comparison principle shown in [26, Proposition 2.2].

**Lemma 4.5** *Let  $\Omega$  be any domain of  $\mathbb{R}^N$ , and  $\tau, \kappa \in (0, \infty)$ ,  $A, B \in \mathbb{R}$ . Let  $U \in C([0, \tau]; L_{loc}^2(\overline{\Omega}))$  such that  $U_t, \nabla U, D^2U \in L_{loc}^2(\overline{\Omega} \times (0, \tau))$ ,  $\text{ess sup}_{Q_{\Omega, \tau}} U < \infty$ ,  $U \leq B$  on the parabolic boundary of  $Q_{\Omega, \tau}$ , and*

$$U_t - \Delta U \leq \kappa(1 + |x|)|\nabla U| + f \quad \text{in } Q_{\Omega, \tau}$$

*where  $f = f(x, t)$  such that  $f(\cdot, t) \in L_{loc}^2(\overline{\Omega})$  for a.e.  $t \in (0, \tau)$  and  $f \leq 0$  on  $\{(x, t) \in Q_{\Omega, \tau} : U(x, t) \geq A\}$ . Then  $\text{ess sup}_{Q_{\Omega, \tau}} U \leq \max(A, B)$ .*

**Proof.** We set  $\varphi(x, t) = \Lambda t + \ln(1 + |x|^2)$ ,  $\Lambda > 0$ . Then  $\nabla \varphi = 2x/(1 + |x|^2)$ ,  $0 \leq \Delta \varphi \leq 2N/(1 + |x|^2) \leq 2N$ . Let  $\varepsilon > 0$  and  $Y = U - \max(A, B) - \varepsilon \varphi$ . Taking  $\Lambda = 2\sqrt{2}\kappa + 2N$ , we obtain

$$Y_t - \Delta Y - f - \kappa(1 + |x|)|\nabla Y| \leq \varepsilon(K(1 + |x|)|\nabla \varphi| - \varphi_t + \Delta \varphi) \leq \varepsilon(2\sqrt{2}\kappa + 2N - \Lambda) = 0.$$

Since  $\text{ess sup}_{Q_{\Omega, \tau}} U < \infty$ , for  $R$  large enough, and any  $t \in (0, \tau)$ , we have  $Y(\cdot, t) \leq 0$  a.e. in  $\Omega \cap \{|x| > R\}$ . And  $Y^+ \in C([0, \tau]; L^2(\Omega)) \cap W^{1,2}((0, \tau); L^2(\Omega))$ ,  $Y^+(0) = 0$  and  $Y^+(\cdot, t) \in W^{1,2}(\Omega \cap B_R)$  for a.e.  $t \in (0, \tau)$ , and  $fY^+(\cdot, t) \leq 0$ . Then

$$\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} Y^{+2}(\cdot, t) \right) \leq - \int_{\Omega} |\nabla Y^+(\cdot, t)|^2 + \kappa(1 + R) \int_{\Omega} |\nabla Y(\cdot, t)| Y^+(\cdot, t) \leq \frac{\kappa^2(1 + R)^2}{4} \int_{\Omega} Y^{+2}(\cdot, t),$$

hence by integration  $Y \leq 0$  a.e. in  $Q_{\Omega, \tau}$ . We conclude as  $\varepsilon \rightarrow 0$ .  $\blacksquare$

**Proof of Theorem 1.2.** We can assume  $x_0 = 0$ . By setting  $u(x, t) = \nu^{q'/2} U(x/\sqrt{\nu}, t)$ , for proving (4.4) we can suppose that  $u$  is a classical solution of (1.1) with  $\nu = 1$ . We set

$$\delta + u = v^{\frac{q}{q-1}}, \quad \delta \in (0, 1).$$

(i) **Local problem relative to  $|\nabla v|^2$ .** Here  $u$  is any classical solution  $u$  of equation (1.1) in a cylinder  $Q_{B_M, T}$  with  $M > 0$ . Then  $v$  satisfies the equation

$$v_t - \Delta v = \frac{1}{q-1} \frac{|\nabla v|^2}{v} - cv|\nabla v|^q, \quad c = (q')^{q-1}. \quad (4.19)$$

Setting  $w = |\nabla v|^2$ , we define

$$\mathcal{L}w = w_t - \Delta w + b \cdot \nabla w, \quad b = (qcvw^{\frac{q-2}{2}} - \frac{2}{q-1} \frac{1}{v}) \nabla v.$$

Differentiating (4.19) and using the identity  $\Delta w = 2\nabla(\Delta v) \cdot \nabla v + 2|D^2v|^2$ , we obtain the equation

$$\mathcal{L}w + 2cw^{\frac{q+2}{2}} + 2|D^2v|^2 + \frac{2}{q-1} \frac{w^2}{v^2} = 0. \quad (4.20)$$

As in [26], for  $s \in (0, 1)$ , we consider a test function  $\zeta \in C^2(\overline{B}_{3M/4})$  with values in  $[0, 1]$ ,  $\zeta = 0$  for  $|x| \geq 3M/4$  and  $|\nabla \zeta| \leq C(N, s)\zeta^s/M$  and  $|\Delta \zeta| + |\nabla \zeta|^2/\zeta \leq C(N, s)\zeta^s/M^2$  in  $B_{3M/4}$ . We set  $z = w\zeta$ . We have

$$\mathcal{L}z = \zeta \mathcal{L}w + w \mathcal{L}\zeta - 2\nabla w \cdot \nabla \zeta \leq \zeta \mathcal{L}w + w \mathcal{L}\zeta + |D^2v|^2 \zeta + 4w \frac{|\nabla \zeta|^2}{\zeta}.$$

It follows that in  $Q_{B_M, T}$ ,

$$\mathcal{L}z + 2cw^{\frac{q+2}{2}}\zeta + \frac{2}{q-1} \frac{w^2}{v^2} \zeta \leq \frac{C\zeta^s w}{M^2} + \frac{C\zeta^s w^{\frac{3}{2}}}{M} \left| cqvw^{\frac{q-2}{2}} - \frac{2}{q-1} \frac{1}{v} \right| \leq C\zeta^s \left( \frac{w}{M^2} + \frac{vw^{\frac{q+1}{2}}}{M} + \frac{w^{\frac{3}{2}}}{Mv} \right),$$

with constants  $C = C(N, q, s)$ . Since  $\zeta \leq 1$ , from the Young inequality, taking  $s \geq \max(q+1, 3)/(q+2)$ , for any  $\varepsilon > 0$ ,

$$\frac{C}{M} \zeta^s vw^{\frac{q+1}{2}} = \frac{C}{M} \zeta^{\frac{q+1}{q+2}} \zeta^{s-\frac{q+1}{q+2}} vw^{\frac{q+1}{2}} \leq \varepsilon \zeta w^{\frac{q+2}{2}} + C(N, q, \varepsilon) \frac{v^{q+2}}{M^{q+2}},$$

and

$$\begin{aligned} \frac{C}{M^2} \zeta^s w &\leq \varepsilon \zeta w^{\frac{q+2}{2}} + C(N, q, \varepsilon) \frac{1}{M^{\frac{2(q+2)}{q}}}, \\ \frac{C}{M} \zeta^s \frac{w^{\frac{3}{2}}}{v} &\leq \frac{1}{\delta M} \zeta^s w^{\frac{3}{2}} = \frac{1}{\delta M} \zeta^{s-\frac{3}{q+2}} \zeta^{\frac{3}{q+2}} w^{\frac{3}{2}} \leq \varepsilon \zeta w^{\frac{q+2}{2}} + C(N, q, \varepsilon) \frac{1}{(\delta M)^{\frac{q+2}{q-1}}}. \end{aligned}$$

Then with a new  $C = C(N, q, \delta)$

$$\mathcal{L}z + cz^{\frac{q+2}{2}} \leq C \left( \frac{v^{q+2}}{M^{q+2}} + \frac{1}{M^{\frac{2(q+2)}{q}}} + \frac{1}{M^{\frac{q+2}{q-1}}} \right). \quad (4.21)$$

**(ii) Nonuniversal estimates of  $w$ .** Here we assume that  $u$  is a classical solution of (1.1) in whole  $Q_{\mathbb{R}^N, T}$ , such that  $u \in C(\mathbb{R}^N \times [0, T])$ . From Theorem 4.4, for any  $\tau \in (0, T)$ , there holds in  $Q_{\mathbb{R}^N, \tau}$

$$v(x, t) = (\delta + u(x, t))^{\frac{q-1}{q}} \leq C(t^{-\frac{1}{q}} |x| + (t + \sup_{B_{2\eta}} u_0)^{\frac{q-1}{q}}), \quad C = C(N, q, \eta, \tau). \quad (4.22)$$

hence for  $M \geq M(q, \sup_{B_{2\eta}} u_0, \tau) \geq 1$ , we deduce

$$v(x, t) \leq 2Ct^{-\frac{1}{q}} M, \quad \text{in } Q_{B_M, \tau}.$$



Then with a new constant  $C = C(N, q, \eta, \tau, \delta)$ , there holds in  $Q_{B_{3M/4}, \tau}$

$$\mathcal{L}z + cz^{\frac{q+2}{2}} \leq Ct^{-\frac{q+2}{q}}. \quad (4.23)$$

Next we consider  $\Psi(t) = Kt^{-2/q}$ . It satisfies

$$\Psi_t + c\Psi^{\frac{q+2}{2}} = (cK^{\frac{q+2}{2}} - 2q^{-1}K)t^{-\frac{q+2}{q}} \geq Ct^{-\frac{q+2}{q}}$$

if  $K \geq \overline{K} = \overline{K}(N, q, \eta, \tau, \delta)$ . Fixing  $\epsilon \in (0, T)$  such that  $\tau + \epsilon < T$ , there exists  $\tau_\epsilon \in (0, \epsilon)$  such that  $\Psi(\theta) \geq \sup_{B_M} z(., \epsilon)$  for any  $\theta \in (0, \tau_\epsilon)$ . We have

$$\begin{aligned} z_t(., t + \epsilon) - \Delta z(., t + \epsilon) + b(., t + \epsilon) \cdot \nabla(z, t + \epsilon) + cz^{\frac{q+2}{2}}(t + \epsilon) \\ \leq C(t + \epsilon)^{-\frac{q+2}{q}} \leq C(t + \theta)^{-\frac{q+2}{q}} \leq \Psi_t(t + \theta) + c\Psi^{\frac{q+2}{2}}(t + \theta). \end{aligned}$$

Therefore, setting  $\tilde{z}(., t) = z(., t + \epsilon) - \Psi(t + \theta)$ , there holds

$$\tilde{z}(., t) - \Delta \tilde{z}(., t) + b(., t + \epsilon) \cdot \nabla \tilde{z}(., t) \leq 0$$

on the set  $\mathcal{V} = \{(x, t) \in Q_{B_{3M/4}, \tau + \epsilon} : \tilde{z}(x, t) \geq 0\}$ ; otherwise  $\tilde{z}(., t) \leq 0$  for sufficiently small  $t > 0$ , and  $\tilde{z} \leq 0$  on  $\partial B_{3M/4} \times [0, \tau]$ . Then from Lemma 4.5, we get  $z(., t + \epsilon) \leq \Psi(t + \theta)$  in  $Q_{B_{3M/4}, \tau}$ , since  $|b| \leq (qcvw^{\frac{q-1}{2}} + \frac{2}{q-1}\frac{1}{\delta}w^{1/2})$ , hence bounded on  $Q_{B_{3M/4}, \tau + \epsilon}$ . Going to the limit as  $\theta, \epsilon \rightarrow 0$ , we deduce that  $z(., t) \leq \overline{K}t^{-\frac{2}{q}}$  in  $Q_{B_{3M/4}, \tau}$ , thus  $w(., t) \leq \overline{K}t^{-\frac{2}{q}}$  in  $Q_{B_{M/2}, \tau}$ . Next we go to the limit as  $M \rightarrow \infty$  and deduce that  $w(., t) \leq \overline{K}t^{-\frac{2}{q}}$  in  $Q_{\mathbb{R}^N, \tau}$ , namely

$$(q')^q |\nabla v(., t)|^q = \frac{|\nabla u|^q}{\delta + u}(., t) \leq Ct^{-1}, \quad C = C(N, q, \eta, \delta, \tau).$$

In turn for any  $\epsilon$  as above, there holds  $w \in L^\infty(Q_{\mathbb{R}^N, \epsilon, T})$ , that means  $|\nabla v| \in L^\infty(Q_{\mathbb{R}^N, \epsilon, \tau})$ .

**(iii) Universal estimate (4.4) for  $u \in C(\mathbb{R}^N \times [0, T])$ :** we prove the universal estimate (4.4). Taking again  $\Psi(t) = Kt^{-2/q}$ , with now  $K = K(N, q) = q^{-2}(q-1)^{2/q'}$ , we have

$$\Psi_t + 2c\Psi^{\frac{q+2}{2}} \geq (2cK^{\frac{q+2}{2}} - 2q^{-1}K)t^{-\frac{q+2}{q}} \geq 0.$$

And  $\mathcal{L}w + 2cw^{\frac{q+2}{2}} \leq 0$  from (4.20). Moreover there exists  $\tau_\epsilon \in (0, \tau)$  such that  $\Psi(\theta) \geq \sup_{\mathbb{R}^N} w(., \epsilon)$  for any  $\theta \in (0, \tau_\epsilon)$ . Setting  $y(., t) = w(., t + \epsilon) - \Psi(., t + \theta)$ , hence on the set  $\mathcal{U} = \{(x, t) \in Q_{\mathbb{R}^N, \tau} : y(x, t) \geq 0\}$ , there holds in the same way

$$y(., t) - \Delta y(., t) + b(., t + \epsilon) \cdot \nabla y(., t) \leq 0.$$

Here we only have from (4.22)

$$|b| \leq (qcvw^{\frac{q-1}{2}} + \frac{2}{q-1}\frac{1}{\delta}w^{1/2}) \leq \kappa_\epsilon(1 + |x|)$$

on  $Q_{\mathbb{R}^N, \epsilon, \tau}$ , for some  $\kappa_\epsilon = \kappa_\epsilon(N, q, \eta, \sup_{B_{2\eta}} u_0, \tau, \epsilon)$ . It is sufficient to apply Lemma 4.5. We deduce that  $w(., t + \epsilon) \leq \Psi(t + \theta)$  on  $(0, \tau)$ . As  $\theta, \epsilon \rightarrow 0$  we obtain that  $w(., t) \leq \Psi(t) = q^{-2}(q-1)^{2/q'} t^{-2/q}$ , which shows now that in  $(0, T)$

$$|\nabla v(., t)|^q = (q')^{-q} \frac{|\nabla u|^q}{\delta + u}(., t) \leq q^{-q}(q-1)^{(q-1)} t^{-1}.$$

As  $\delta \rightarrow 0$ , we obtain (4.4).

**(iv) General universal estimate.** Here we relax the assumption  $u \in C(\mathbb{R}^N \times [0, T])$  : For any  $\epsilon \in (0, T)$  such that  $\tau + \epsilon < T$ , we have  $u \in C(\mathbb{R}^N \times [\epsilon, \tau + \epsilon])$ , then from above,

$$|\nabla v(., t + \epsilon)|^q \leq \frac{1}{q-1} \frac{1}{t},$$

and we obtain (4.4) as  $\epsilon \rightarrow 0$ , on  $(0, \tau)$  for any  $\tau < T$ , hence on  $(0, T)$ . ■

**Proof of Theorem 1.3.** It is a direct consequence of Theorems 1.2 and 4.1. ■

## 5 Existence and nonuniqueness results

First mention some known uniqueness and comparison results, for the Cauchy problem, see [11, Theorems 2.1, 4.1, 4.2 and Remark 2.1 ], [13, Theorem 2.3, 4.2, 4.25, Proposition 4.26 ], and for the Dirichlet problem, see [1, Theorems 3.1, 4.2], [6], [13, Proposition 5.17], [24].

**Theorem 5.1** *Let  $\Omega = \mathbb{R}^N$  (resp.  $\Omega$  bounded). (i) Let  $1 < q < q_*$ , and  $u_0 \in \mathcal{M}_b(\mathbb{R}^N)$  (resp.  $u_0 \in \mathcal{M}_b(\Omega)$ ). Then there exists a unique weak solution  $u$  of (1.1) with trace  $u_0$  (resp. a weak solution of  $(D_{\Omega, T})$ , such that  $\lim_{t \rightarrow 0} u(., t) = u_0$  weakly in  $\mathcal{M}_b(\Omega)$ ). If  $v_0 \in \mathcal{M}_b(\Omega)$  and  $u_0 \leq v_0$ , and  $v$  is the solution associated to  $v_0$ , then  $u \leq v$ .*

*(ii) Let  $u_0 \in L^R(\Omega)$ ,  $1 \leq R \leq \infty$ . If  $1 < q < (N+2R)/(N+R)$ , or if  $q = 2$ ,  $R < \infty$ , there exists a unique weak solution  $u$  of (1.1) (resp.  $(D_{\Omega, T})$ ) such that  $u \in C([0, T]; L^R(\Omega))$  and  $u(0) = u_0$ . If  $v_0 \in L^R(\mathbb{R}^N)$  and  $u_0 \leq v_0$ , then  $u \leq v$ . If  $u_0$  is nonnegative, then for any  $1 < q \leq 2$ , there still exists at least a weak nonnegative solution  $u$  satisfying the same conditions.*

Next we prove Theorem 1.4. Our proof of (ii) (iii) is based on approximations by nonincreasing sequences. Another proof can be obtained when  $u_0 \in L^1_{loc}(\mathbb{R}^N)$  and  $q \leq 2$ , by techniques of equiintegrability, see [22] for a connected problem.

**Proof of Theorem 1.4.** Assume  $\Omega = \mathbb{R}^N$  (resp.  $\Omega$  bounded).

(i) Case  $1 < q < q_*$ ,  $u_0 \in \mathcal{M}^+(\mathbb{R}^N)$  (resp.  $\mathcal{M}^+(\Omega)$ ): Let  $u_{0,n} = u_0 \mathbf{1}_{B_n}$  (resp.  $u_{0,n} = u_0 \mathbf{1}_{\overline{\Omega'_{1/n}}}$ , where  $\Omega_n = \{x \in \Omega : d(x, \partial\Omega) > 1/n\}$ , for  $n$  large enough). From Theorem 5.1, there exists a unique weak solution  $u_n$  of (1.1) (resp. of  $(D_{\Omega, T})$ ) with trace  $u_{0,n}$ , and  $(u_n)$  is nondecreasing; and  $u_n \in C^{2,1}(Q_{\mathbb{R}^N, T})$  since  $q \leq 2$ . From (3.1), (3.5), for any  $\xi \in C^1_c(\Omega)$ ,

$$\int_{\Omega} u_n(., t) \xi^{q'} + \frac{1}{2} \int_0^t \int_{\Omega} |\nabla u_n|^q \xi^{q'} \leq Ct \int_{\Omega} |\nabla \xi|^{q'} + \int_{\Omega} \xi^{q'} du_0. \quad (5.1)$$

Hence  $(u_n)$  is bounded in  $L_{loc}^\infty([0, T]; L_{loc}^1(\Omega))$ , and  $(|\nabla u_n|^q)$  is bounded in  $L_{loc}^1([0, T]; L_{loc}^1(\Omega))$ . In turn  $(u_n)$  is bounded in  $L_{loc}^\infty((0, T); L_{loc}^\infty(\Omega))$ , from Theorem 3.3. From Theorem 2.3, up to a subsequence,  $(u_n)$  converges in  $C_{loc}^{2,1}(Q_{\mathbb{R}^N, T})$  (resp.  $C_{loc}^{2,1}(Q_{\Omega, T}) \cap C^{1,0}(\overline{\Omega} \times (0, T))$ ) to a weak solution  $u$  of (1.1) in  $Q_{\mathbb{R}^N, T}$  (resp. of  $(D_{\Omega, T})$ ). Also from [3, Lemma 3.3], for any  $k \in [1, q^*)$  and any  $0 < s < \tau < T$ ,

$$\|u_n\|_{L^k((s, \tau); W^{1, k}(\omega))} \leq C(k, \omega)(\|u_n(s, \cdot)\|_{L^1(\omega)} + \| |\nabla u_n|^q + |\nabla u_n| + u_n \|_{L^1(Q_{\omega, s, \tau})}), \quad \forall \omega \subset \subset \Omega$$

$$(\text{resp. } \|u_n\|_{L^k((s, \tau); W_0^{1, k}(\Omega))} \leq C(k, \Omega)(\|u_n(\cdot, s)\|_{L^1(\Omega)} + \| |\nabla u_n|^q + |\nabla u_n| + u_n \|_{L^1(Q_{\Omega, s, \tau})})).$$

hence  $(u_n)$  is bounded in  $L_{loc}^k([0, T]; W_{loc}^{1, k}(\mathbb{R}^N))$  (resp.  $L_{loc}^k([0, T]; W_0^{1, k}(\Omega))$ ). Since  $q < q_*$ ,  $(|\nabla u_n|^q)$  is equiintegrable in  $Q_{B_M, \tau}$  for any  $M > 0$  (resp. in  $Q_{\Omega, \tau}$ ) and  $\tau \in (0, T)$ , then  $(|\nabla u|^q) \in L_{loc}^1([0, T]; L_{loc}^1(\Omega))$ . From (2.6),

$$\int_{\Omega} u_n(t, \cdot) \xi + \int_0^t \int_{\Omega} |\nabla u_n|^q \xi = - \int_0^t \int_{\Omega} \nabla u_n \cdot \nabla \xi + \int_{\Omega} \xi du_0. \quad (5.2)$$

As  $n \rightarrow \infty$  we obtain

$$\int_{\Omega} u(t, \cdot) \xi + \int_0^t \int_{\Omega} |\nabla u|^q \xi = - \int_0^t \int_{\Omega} \nabla u \cdot \nabla \xi + \int_{\Omega} \xi du_0.$$

Thus  $\lim_{t \rightarrow 0} \int_{\Omega} u(\cdot, t) \xi = \int_{\Omega} \xi du_0$ , for any  $\xi \in C_c^{1+}(\Omega)$ , hence for any  $\xi \in C_c^+(\Omega)$ ; hence  $u$  admits the trace  $u_0$ .

(ii) Case  $q_* \leq q \leq 2$ . Let us set  $u_{0,n} = \min(u_0, n) \chi_{B_n}$  (resp.  $u_{0,n} = \min(u_0, n) \chi_{\overline{\Omega}_{1/n}}$  for  $n$  large enough). Then  $u_{0,n} \in L^R(\Omega)$  for any  $R \geq 1$ . From Theorem 5.1, the problem admits a solution  $u_n$ , and it is unique in  $C([0, T]; L^R(\Omega))$  for any  $R > (2 - q)/N(q - 1)$  and then  $(u_n)$  is nondecreasing. As above,  $(u_n)$  is bounded in  $L_{loc}^\infty([0, T]; L_{loc}^1(\Omega))$ ,  $(|\nabla u_n|^q)$  is bounded in  $L_{loc}^1([0, T]; L_{loc}^1(\Omega))$ ,  $(u_n)$  is bounded in  $L_{loc}^\infty((0, T); L_{loc}^\infty(\Omega))$  from Theorem 3.3. From Theorem 2.3,  $(u_n)$  converges in  $C_{loc}^{2,1}(Q_{\Omega, T})$  to a weak solution  $u$  of (1.1) in  $Q_{\Omega, T}$ , such that  $u \in L_{loc}^\infty([0, T]; L_{loc}^1(\Omega))$  and  $|\nabla u|^q \in L_{loc}^1([0, T]; L_{loc}^1(\Omega))$ .

Then from Remark 2.5,  $u$  admits a trace  $\mu_0 \in \mathcal{M}^+(\Omega)$  as  $t \rightarrow 0$ . Applying (5.2) to  $u_n$ , since  $u_n \leq u$ , we get

$$\lim_{t \rightarrow 0} \int_{\Omega} u(\cdot, t) \xi = \int_{\Omega} \xi d\mu_0 \geq \lim_{t \rightarrow 0} \int_{\Omega} u_n(\cdot, t) \xi = \int_{\Omega} \xi du_0,$$

for any  $\xi \in C_c^+(\Omega)$ ; thus  $u_0 \leq \mu_0$ . Moreover

$$\int_{\Omega} u_n(t, \cdot) \xi + \int_0^t \int_{\Omega} |\nabla u_n|^q \xi = \int_0^t \int_{\Omega} u_n \Delta \xi dx + \int_{\Omega} \xi du_0.$$

And  $(u_n)$  is bounded in  $L^k(Q_{\omega, \tau})$  for any  $k \in (1, q_*)$ ; then for any domain  $\omega \subset \subset \Omega$ ,  $(u_n)$  converges strongly in  $L^1(Q_{\omega, \tau})$ ; then from the convergence *a.e.* of the gradients, and the Fatou Lemma,

$$\int_{\mathbb{R}^N} u(t, \cdot) \xi + \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q \xi \leq \int_0^t \int_{\mathbb{R}^N} u \Delta \xi dx + \int_{\mathbb{R}^N} \xi du_0.$$

But from Remark 2.5,

$$\int_{\mathbb{R}^N} u(t, \cdot) \xi + \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q \xi = \int_0^t \int_{\mathbb{R}^N} u \Delta \xi dx + \int_{\mathbb{R}^N} \xi d\mu_0,$$

then  $\mu_0 \leq u_0$ , hence  $\mu_0 = u_0$ . Finally we prove the continuity: Let  $\xi \in \mathcal{D}^+(\Omega)$  and  $\omega \subset\subset \Omega$  containing the support of  $\xi$ . Then  $z = u\xi$  is solution of the Dirichlet problem

$$\begin{cases} z_t - \Delta z = g, & \text{in } Q_{\omega,T}, \\ z = 0, & \text{on } \partial\omega \times (0,T), \\ \lim_{t \rightarrow 0} z(\cdot, t) = \xi u_0, & \text{weakly in } \mathcal{M}_b(\omega), \end{cases}$$

with  $g = -|\nabla u|^q \xi + v(-\Delta \psi) - 2\nabla v \cdot \nabla \psi \in L^1(Q_{\omega,T})$ . The solution is unique, see [6, Proposition 2.2]. Since  $u_0 \in L^1_{loc}(\Omega)$ , there also exists a unique solution such that  $z \in C([0,T], L^1(\omega))$  from [3, Lemma 3.3], hence  $u \in C([0,T], L^1_{loc}(\Omega))$ .

(iii) Case  $q > 2$ . We get the existence as above, by taking for  $(u_{0,n})$  a nondecreasing sequence in  $C_b(\mathbb{R}^N)$  (resp. in  $C_0(\Omega)$ ), converging to  $u_0$ , and using Remark 2.4 for classical solutions. ■

Next we show the nonuniqueness of the weak solutions when  $q > 2$ : here the coefficient  $a$  defined at (1.7) is negative, and  $|a| = (q-2)/(q-1) < 1$ .

**Proof of Theorem 1.5.** Since  $q > 2$  and  $N \geq 2$ , the function  $\tilde{U}$  is a solution in  $\mathcal{D}'(\mathbb{R}^N)$  of the stationary equation

$$-\Delta u + |\nabla u|^q = 0$$

Indeed  $\tilde{U} \in W^{1,q}_{loc}(\mathbb{R}^N) \cap W^{2,1}_{loc}(\mathbb{R}^N)$  because  $N > q'$ , and  $\tilde{U}$  is a classical solution in  $\mathbb{R}^N \setminus \{0\}$ . Then it is a weak solution of  $(P_{\mathbb{R}^N, \infty})$ , and  $\tilde{U} \notin C^1(Q_{\mathbb{R}^N, \infty})$ . Since  $\tilde{U} \in C(\mathbb{R}^N)$ , from Theorem ??, or from [5], there exists also a classical solution  $U_{\tilde{C}} \in C^{2,1}(Q_{\mathbb{R}^N, \infty})$  of the problem, thus  $U_{\tilde{C}} \neq U_0$ .

More generally, for any  $C > 0$ , there exists a classical solution  $U_C$  with trace  $C|x|^{|a|}$ . And  $U_C$  is obtained as the limit of the nondecreasing sequence of the unique solutions  $U_{n,C}$  with trace  $\min(C|x|^{|a|}, n)$ , then it is radial. Moreover for any  $\lambda > 0$ , the function  $U_{n,C,\lambda}(x,t) = \lambda^{-a}U_{n,C}(\lambda x, \lambda^2 t)$  admits the trace  $\min(C|x|^{|a|}, n\lambda^{-a})$ . Therefore, denoting by  $k_{\lambda,n}$  the integer part of  $n\lambda^{-a}$ , there holds  $U_{k_{\lambda,n},C} \leq U_{n,C,\lambda} \leq U_{k_{\lambda,n}+1,C}$  from the comparison principle. And  $U_{n,C,\lambda}(x,t)$  converges everywhere to  $\lambda^{-a}U_C(\lambda x, \lambda^2 t)$ , thus  $U_C(x,t) = \lambda^{-a}U_C(\lambda x, \lambda^2 t)$ , that means  $U_C$  is self-similar. Then  $U_C$  has the form (1.14), where  $f \in C^2([0, \infty))$ ,  $f(0) \geq 0$ ,  $f'(0) = 0$ ,  $\lim_{\eta \rightarrow \infty} \eta^{-|a|/2} f(\eta) = C$ , and for any  $\eta > 0$ ,

$$f''(\eta) + \left(\frac{N-1}{\eta} + \frac{\eta}{2}\right)f'(\eta) - \frac{|a|}{2}f(\eta) - |f'(\eta)|^q = 0. \quad (5.3)$$

From the Cauchy-Lipschitz Theorem, we find  $f(0) > 0$ , since  $f \not\equiv 0$ , hence  $f''(0) > 0$ . The function  $f$  is increasing: indeed if there exists a first point  $\eta_0 > 0$  such that  $f'(\eta_0) = 0$ , then  $f''(\eta_0) > 0$ , which is contradictory. ■

## 6 Second local regularizing effect

Here we show the second regularizing effect. We prove an estimate, playing the role of the sub-caloricity estimate (2.4). Our proof follows the general scheme of Stampacchia's method, developped by many authors, see [17] and references there in, and [19].

First we write estimate (3.1) in another form, and from Gagliardo estimate, we obtain the following:

**Lemma 6.1** *Let  $q > 1$ . Let  $\eta > 0, r \geq 1$ . Let  $u$  be any nonnegative weak subsolution of equation (1.1) in  $Q_{\Omega, T}$ . Let  $B_{2\eta} \subset\subset \Omega$ ,  $0 < \theta < \tau < T$ , and  $\xi \in C^1((0, T), C_c^1(\Omega))$ , with values in  $[0, 1]$ , such that  $\xi(., t) = 0$  for  $t \leq \theta$ . Let  $\lambda \geq \max(2, q')$ .*

*Then for any  $\nu \in (0, 1]$ ,*

$$\sup_{[\theta, \tau]} \int_{\Omega} u^r(., t) \xi^{\lambda} + \frac{\int_{\theta}^{\tau} \int_{\Omega} u^{(q+r-1)(1+\frac{\mu}{N})} \xi^{\lambda(1+\frac{\mu}{N})}}{(\sup_{t \in [\theta, \tau]} \int_{\Omega} u^r \xi^{\frac{\lambda r}{q+r-1}})^{\frac{q}{N}}} \leq C \int_{\theta}^{\tau} \int_{\Omega} (u^r |\xi_t| + u^{r-1} |\nabla \xi|^{q'} + u^{q+r-1} |\nabla \xi|^q), \quad (6.1)$$

where  $\mu = rq/(q+r-1)$ ,  $C = C(N, q, r, \lambda)$ .

**Proof.** From Remark 2.2,  $u \in L_{loc}^{\infty}(Q_{\Omega, T})$ , and hence  $u^{\frac{q+r-1}{q}} \xi^{\frac{\lambda}{q}} \in W^{1, q}(Q_{\Omega, \theta, t})$  and

$$\begin{aligned} \int_{\theta}^t \int_{\Omega} |\nabla(u^{\frac{q+r-1}{q}} \xi^{\frac{\lambda}{q}})|^q &= \int_{\theta}^t \int_{\Omega} \left| \frac{q+r-1}{q} u^{\frac{r-1}{q}} \xi^{\frac{\lambda}{q}} \nabla u + \frac{\lambda}{q} u^{\frac{q+r-1}{q}} \xi^{\frac{\lambda-q}{q}} \nabla \xi \right|^q \\ &\leq C \left( \int_{\theta}^t \int_{\Omega} u^{r-1} |\nabla u|^q \xi^{\lambda} + \int_{\theta}^t \int_{\Omega} u^{q+r-1} |\nabla \xi|^q \xi^{\lambda-q} \right), \end{aligned}$$

with  $C = C(q, r, \lambda)$ . From (3.1), since  $\nu \leq 1$ , we get

$$\sup_{[\theta, \tau]} \int_{\Omega} u^r(., t) \xi^{\lambda} + \int_{\theta}^{\tau} \int_{\Omega} |\nabla(u^{\frac{q+r-1}{q}} \xi^{\frac{\lambda}{q}})|^q \leq C \int_{\theta}^{\tau} \int_{\Omega} (u^r |\xi_t| + u^{r-1} |\nabla \xi|^{q'} + u^{q+r-1} |\nabla \xi|^q), \quad (6.2)$$

where  $C = C(q, r, \lambda)$ . Next we use a Galliaro type estimate, see [17, Proposition 3.1]: for any  $\mu \geq 1$ , and any  $w \in L_{loc}^{\infty}((0, T), L^{\mu}(\Omega)) \cap L_{loc}^q((0, T), W^{1, q}(\Omega))$ ,

$$\int_{\theta}^{\tau} \int_{\Omega} w^{q(1+\frac{\mu}{N})} \leq C \left( \int_{\theta}^{\tau} \int_{\Omega} |\nabla w|^q \right) \left( \sup_{t \in [\theta, \tau]} \int_{\Omega} |w|^{\mu} \right)^{\frac{q}{N}}, \quad C = C(N, q, \mu).$$

Taking  $w = u^{\frac{q+r-1}{q}} \xi^{\frac{\lambda}{q}}$  and  $\mu = qr/(q+r-1) \geq r \geq 1$ , setting  $s = 1 + \mu/N$ , it comes

$$\int_{\theta}^{\tau} \int_{\Omega} u^{(q+r-1)s} \xi^{\lambda s} \leq C \left( \int_{\theta}^{\tau} \int_{\Omega} |\nabla w|^q \right) \left( \sup_{t \in [\theta, \tau]} \int_{\Omega} u^r \xi^{\frac{\lambda r}{q+r-1}} \right)^{\frac{q}{N}},$$

hence (6.1) follows. ■

**Theorem 6.2** *Let  $q > 1$ . Let  $u$  be any nonnegative weak solution of equation (1.1) in  $Q_{\Omega, T}$ . Let  $B(x_0, \rho) \subset\subset \Omega$ . Let  $R > q-1$  (in particular any  $R \geq 1$  if  $q < 2$ ). Then there exists  $C = C(N, q, R)$  such that, for any  $t, \theta$  such that  $0 < t - 2\theta < t < T$ ,*

$$\begin{aligned} \sup_{B(x_0, \frac{\rho}{2}) \times [t-\theta, t]} u &\leq C \theta^{-\frac{N+q}{qR+N(q-1)}} \left( \int_{t-2\theta}^t \int_{B(x_0, \rho)} u^R \right)^{\frac{q}{qR+N(q-1)}} \\ &\quad + C \rho^{-\frac{N+q}{(q-1)(R+N+1)}} \left( \int_{t-2\theta}^t \int_{B(x_0, \rho)} u^R \right)^{\frac{1}{R+N+1}} + C \rho^{-\frac{N+q}{R+1-q}} \left( \int_{t-2\theta}^t \int_{B(x_0, \rho)} u^R \right)^{\frac{1}{R+1-q}}. \end{aligned} \quad (6.3)$$

**Proof.** Since  $u \in C((0, T); L_{loc}^R(Q_{\Omega, T}))$ , by regularization we can assume that  $u$  is a classical solution in  $Q_{\Omega, T}$ . Let  $t, \theta$  such that  $0 < t - 2\theta < t < T$ . We can assume  $x_0 = 0 \in \Omega$ . By translation of  $t - \theta$ , we are lead to prove that for any solution in  $Q_{\Omega, -\tau/2, \tau/2}$  ( $\tau < T$ ),

$$\begin{aligned} \sup_{Q_{B_{\rho/2}, 0, \theta}} u &\leq C\theta^{-\frac{N+q}{qR+N(q-1)}} \left( \int_{-\theta}^{\theta} \int_{B_{\rho}} u^R \right)^{\frac{q}{qR+N(q-1)}} \\ &\quad + C\rho^{-\frac{N+q}{(q-1)(R+N+1)}} \left( \int_{-\theta}^{\theta} \int_{B_{\rho}} u^R \right)^{\frac{1}{R+N+1}} + C\rho^{-\frac{N+q}{R+1-q}} \left( \int_{-\theta}^{\theta} \int_{B_{\rho}} u^R \right)^{\frac{1}{R+1-q}}. \end{aligned} \quad (6.4)$$

For given  $k > 0$  we set  $u_k = (u - k)^+$ . Then  $u_k \in C(0, T); L_{loc}^R(Q_{\Omega, T})$ , and  $u_k$  is a weak subsolution of equation (1.1), from the Kato inequality. We set

$$\begin{aligned} \rho_n &= (1 + 2^{-n})\rho/2, & t_n &= -(1 + 2^{-n})\theta/2, \\ Q_n &= B_{\rho_n} \times (t_n, \theta), & Q_0 &= B_{\rho} \times (-\theta, \theta), & Q_{\infty} &= B_{\rho/2} \times (-\theta/2, \theta), \\ k_n &= (1 - 2^{-(n+1)})k, & \tilde{k} &= (k_n + k_{n+1})/2. \end{aligned}$$

and set  $M_{\sigma} = \sup_{Q_{\infty}} u$ ,  $M = \sup_{Q_0} u$ . Let  $\xi(x, t) = \xi_1(x)\xi_2(t)$  where  $\xi_1 \in C_c^1(\Omega)$ ,  $\xi_2 \in C^1(\mathbb{R})$ , with values in  $[0, 1]$ , such that

$$\begin{aligned} \xi_1 &= 1 \quad \text{on } B_{\rho_{n+1}}, & \xi_1 &= 0 \quad \text{on } \mathbb{R}^N \setminus B_{\rho_n}, & |\nabla \xi_1| &\leq C(N)2^{n+1}/\rho; \\ \xi_2 &= 1 \quad \text{on } [t_{n+1}, \infty), & \xi_2 &= 0 \quad \text{on } (-\infty, \theta_n], & |\xi_{2,t}| &\leq C(N)2^{n+1}/\theta. \end{aligned}$$

From Lemma 6.1 we get, with  $\mu = qr/(q + r - 1)$ ,

$$\begin{aligned} \sup_{t \in [t_{n+1}, \theta]} \int_{B_{\rho_{n+1}}} u_{k_{n+1}}^r(\cdot, t) + \frac{\int_{t_{n+1}}^{\theta} \int_{B_{\rho_{n+1}}} u_{k_{n+1}}^{(q+r-1)(1+\frac{\mu}{N})}}{(\sup_{t \in [t_n, \theta]} \int_{B_{\rho_n}} u_{k_n}^r)^{\frac{q}{N}}} &\leq CX_n, \text{ where} \\ X_n &= \int_{t_n}^{\theta} \int_{B_{\rho_n}} (u_{k_{n+1}}^r |\zeta_t| + u_{k_{n+1}}^{r-1} |\nabla \xi|^{q'} + u_{k_{n+1}}^{q+r-1} |\nabla \xi|^q). \end{aligned}$$

Let us define

$$Y_n = \int_{t_n}^{\theta} \int_{B_{\rho_n}} u_{k_n}^{q+r-1}, \quad Z_n = \sup_{t \in [t_n, \theta]} \int_{B_{\rho_n}} u_{k_n}^r, \quad W_n = \int_{t_n}^{\theta} \int_{B_{\rho_n}} \chi_{\{u \geq k_n\}}.$$

Thus, from the Hölder inequality,

$$Z_{n+1} + Z_n^{-\frac{q}{N}} W_{n+1}^{-\frac{\mu}{N}} Y_{n+1}^{1+\frac{\mu}{N}} \leq CX_n. \quad (6.5)$$

Moreover, for any  $\gamma, \beta > 0$ ,

$$\begin{aligned} \int_{t_n}^{\theta} \int_{B_{\rho_n}} u_{k_n}^{\gamma+\beta} &\geq \int_{t_n}^{\theta} \int_{B_{\rho_n}} (k_n - k_{n+1})^{\gamma+\beta} \chi_{\{u \geq k_{n+1}\}} \\ &\geq (k2^{-(n+2)})^{\gamma+\beta} \int_{t_n}^{\theta} \int_{B_{\rho_n}} \chi_{\{u \geq k_{n+1}\}} \geq (k2^{-(n+2)})^{\gamma+\beta} \int_{t_{n+1}}^{\theta} \int_{B_{\rho_{n+1}}} \chi_{\{u \geq k_{n+1}\}}, \end{aligned}$$

and from the Hölder inequality,

$$\begin{aligned}
\int_{t_n}^{\theta} \int_{B_{\rho_n}} u_{k_{n+1}}^{\gamma} &\leq \left( \int_{t_n}^{\theta} \int_{B_{\rho_n}} u_{k_{n+1}}^{\gamma+\beta} \right)^{\frac{\gamma}{\gamma+\beta}} \left( \int_{t_n}^{\theta} \int_{B_{\rho_n}} \chi_{\{u \geq k_{n+1}\}} \right)^{\frac{\beta}{\gamma+\beta}} \\
&\leq \left( \int_{t_n}^{\theta} \int_{B_{\rho_n}} u_{k_n}^{\gamma+\beta} \right) (k^{-1} 2^{(n+2)})^{\beta} \left( \int_{t_n}^{\theta} \int_{B_{\rho_n}} u_{k_n}^{\gamma+\beta} \right)^{\frac{\beta}{\gamma+\beta}} \\
&\leq (k^{-1} 2^{(n+2)})^{\beta} \int_{t_n}^{\theta} \int_{B_{\rho_n}} u_{k_n}^{\gamma+\beta}.
\end{aligned}$$

Thus in particular

$$W_{n+1} \leq C \left( \frac{2^{n+1}}{k} \right)^{q+r-1} Y_n, \quad \int_{t_n}^{\theta} \int_{B_{\rho_n}} u_{k_{n+1}}^r \leq C \left( \frac{2^{n+1}}{k} \right)^{q-1} Y_n, \quad \int_{t_n}^{\theta} \int_{B_{\rho_n}} u_{k_{n+1}}^{r-1} \leq C \left( \frac{2^{n+1}}{k} \right)^q Y_n. \quad (6.6)$$

Otherwise

$$X_n \leq \int_{t_n}^{\theta} \int_{B_{\rho_n}} (2^{n+1} \theta^{-1} u_{k_{n+1}}^r + 2^{q'(n+1)} \rho^{-q'} u_{k_{n+1}}^{r-1} + 2^{q(n+1)} \rho^{-q} u_{k_{n+1}}^{q+r-1}),$$

then from (6.6),

$$X_n \leq C b_0^n f(\theta, \rho, k) Y_n, \quad \text{where } f(\theta, \rho, k) = (\theta^{-1} \frac{1}{k^{q-1}} + \frac{1}{k^q} \rho^{-q'} + \rho^{-q}). \quad (6.7)$$

for some  $b_0$  depending on  $q, r$ . Then from (6.5), (6.6) and (6.7),

$$Z_{n+1} \leq C b_0^n f(\theta, \rho, k) Y_n, \quad Y_{n+1}^{1+\frac{\mu}{N}} \leq C Z_n^{\frac{q}{N}} \left( \frac{2^{n+1}}{k} \right)^{(q+r-1)\frac{\mu}{N}} b_0^n f(\theta, \rho, k) Y_n^{1+\frac{\mu}{N}}.$$

Since  $Y_{n+1} \leq Y_n$ , setting  $\alpha = q/(N + \mu)$  and denoting by  $b_1, b$  some new constants depending on  $N, q, r$ ,

$$\begin{aligned}
Y_{n+2} &\leq C Z_{n+1}^{\frac{q}{N+\mu}} b_1^{n+1} k^{-(q+r-1)\frac{\mu}{N+\mu}} f^{\frac{N}{N+\mu}}(\theta, \rho, k) Y_{n+1} \\
&\leq C (b_0^n f(\theta, \rho, k) Y_n)^{\frac{q}{N+\mu}} b_1^{n+1} k^{-(q+r-1)\frac{\mu}{N+\mu}} f^{\frac{N}{N+\mu}}(\theta, \rho, k) Y_n \\
&\leq C b^n f^{\frac{N+q}{N+\mu}} k^{-(q+r-1)\frac{\mu}{N+\mu}} Y_n^{1+\frac{q}{N+\mu}} := D b^n Y_n^{1+\alpha}.
\end{aligned}$$

From [17, Lemma 4.1],  $Y_n \rightarrow 0$  if

$$Y_0^{\alpha} \delta^{1/\alpha} \leq D^{-1} = C^{-1} k^{(q+r-1)\frac{\mu}{N+\mu}} f^{-\frac{N+q}{N+\mu}},$$

that means

$$k^{qr} \geq c Y_0^q \left( (\theta^{-1} \frac{1}{k^{q-1}} + \frac{1}{k^q} \rho^{-q'} + \rho^{-q}) \right)^{N+q}. \quad (6.8)$$

For getting (6.8) it is sufficient that

$$k^{qr+(q-1)(N+q)} \geq \frac{c}{2} Y_0^q \theta^{-(N+q)}, \quad k^{(r+N+q)} \geq \left( \frac{c}{2} \right)^{1/q} Y_0 \rho^{-\frac{N+q}{q-1}}, \quad \text{and } k^r \geq \frac{c}{2} Y_0 \rho^{-(N+q)}.$$

Thus we deduce that

$$\begin{aligned} \sup_{Q_\infty} u &\leq C\theta^{-\frac{N+q}{qr+(N+q)(q-1)}} \left( \int_{-\theta}^{\theta} \int_{B_\rho} u^{q+r-1} \right)^{\frac{q}{qr+(N+q)(q-1)}} \\ &\quad + C\rho^{-\frac{N+q}{(q-1)(r+N+q)}} \left( \int_{-\theta}^{\theta} \int_{B_\rho} u^{q+r-1} \right)^{\frac{1}{r+N+q}} + C\rho^{-\frac{N+q}{r}} \left( \int_{-\theta}^{\theta} \int_{B_\rho} u^{q+r-1} \right)^{\frac{1}{r}}. \end{aligned} \quad (6.9)$$

If we set  $q + r - 1 = R$ , we obtain (6.4) for any  $R \geq q$ .

Next we consider the case  $R < q$ . From (6.9) we get

$$\begin{aligned} \sup_{B_{\sigma\rho} \times (-\theta/2, \theta)} u &\leq C\theta^{-\frac{N+q}{q+(q-1)(N+q)}} \left( \int_0^\theta \int_{B_\rho} u^q \right)^{\frac{q}{q+(q-1)(N+q)}} \\ &\quad + C\rho^{-\frac{N+q}{(q-1)(1+N+q)}} \left( \int_{-\theta}^\theta \int_{B_\rho} u^q \right)^{\frac{1}{1+N+q}} + C\rho^{-(N+q)} \int_{-\theta}^\theta \int_{B_\rho} u^q \\ &\leq C\theta^{-\frac{N+q}{q+(q-1)(N+q)}} \left( \sup_{B_\rho \times 0, \theta} u \right)^{\frac{q(q-R)}{q+(q-1)(N+q)}} \left( \int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{q}{q+(q-1)(N+q)}} \\ &\quad + C\rho^{-\frac{N+q}{(q-1)(1+N+q)}} \left( \sup_{B_\rho \times 0, \theta} u \right)^{\frac{q(q-R)}{1+N+q}} \left( \int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{1}{1+N+q}} \\ &\quad + C\rho^{-(N+q)} \left( \sup_{B_\rho \times 0, \theta} u \right)^{(q-R)} \int_{-\theta}^\theta \int_{B_\rho} u^R. \end{aligned}$$

We define

$$\tilde{\rho}_n = (1 + 2^{-(n+1)})\rho, \quad \theta_n = -(1 + 2^{-(n+1)})\theta, \quad \tilde{Q}_n = B_{\tilde{\rho}_n} \times (\theta_n, \theta), \quad M_n = \sup_{\tilde{Q}_n} u,$$

hence  $M_0 = \sup_{B_{\rho/2} \times (-\theta/2, \theta)} u$ . We find

$$\begin{aligned} M_n &\leq C\theta^{-\frac{N+q}{q+(q-1)(N+q)}} M_{n+1}^{\frac{q(q-R)}{q+(q-1)(N+q)}} \left( \int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{q}{q+(q-1)(N+q)}} \\ &\quad + C\rho^{-\frac{N+q}{(q-1)(1+N+q)}} M_{n+1}^{\frac{q(q-R)}{1+N+q}} \left( \int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{1}{1+N+q}} + C\rho^{-(N+q)} M_{n+1}^{q-R} \int_{-\theta}^\theta \int_{B_\rho} u^R. \end{aligned}$$

We set

$$\begin{aligned} I &= C\theta^{-\frac{N+q}{q+(q-1)(N+q)}} \left( \int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{q}{q+(q-1)(N+q)}}, \\ J &= C\rho^{-(N+q)} \int_{-\theta}^\theta \int_{B_\rho} u^R, \quad L = C\rho^{-\frac{N+q}{(q-1)(1+N+q)}} \left( \int_{-\theta}^\theta \int_{B_\rho} u^R \right)^{\frac{1}{1+N+q}}. \end{aligned}$$

Note that  $R > q - 1$ , that means  $q - R < 1$ . Then from Hölder inequality,

$$M_n \leq \frac{1}{2} M_{n+1} + C(I^\sigma + L^\delta + J^{\frac{1}{R+1-q}}), \quad \sigma = \frac{q + (q-1)(N+q)}{N(q-1) + qR}, \quad \delta = \frac{1 + N + q}{R + N + 1}.$$



Thus  $M_0 \leq 2^{-n}M_n + 2C(I^\sigma + L^\delta + J^{\frac{1}{R+1-q}})$ , and finally

$$\begin{aligned} M_0 = \sup_{Q_0} u &\leq C(I^\sigma + L^\delta + J^{\frac{1}{R+1-q}}) = C\theta^{-\frac{N+q}{N(q-1)+qR}} \left( \int_{-\theta}^{\theta} \int_{B_\rho} u^R \right)^{\frac{q}{N(q-1)+qR}} \\ &+ C\rho^{-\frac{N+q}{(q-1)(R+N+1)}} \left( \int_{-\theta}^{\theta} \int_{B_\rho} u^R \right)^{\frac{1}{R+N+1}} + C\rho^{-\frac{N+q}{R+1-q}} \left( \int_{-\theta}^{\theta} \int_{B_\rho} u^R \right)^{\frac{1}{R+1-q}}, \end{aligned}$$

which shows again (6.4). Then (6.4) holds for any  $R > q - 1$ , in particular for any  $R \geq 1$  if  $q < 2$ . ■

Now we prove our second regularizing effect due to the effect of the gradient:

**Proof of Theorem 1.6.** We assume  $x_0 = 0$ . Let  $\kappa > 0$  be a parameter. From (6.3), for any  $\rho \in (0, \eta)$  such that  $\rho^\kappa \leq t < \tau$ ,

$$\begin{aligned} \sup_{B_{\frac{\rho}{2}} \times [t-\rho^\kappa, t]} u &\leq C\rho^{-\frac{\kappa(N+q)}{qR+N(q-1)}} \left( \int_{t-\rho^\kappa}^t \int_{B_\rho} u^R \right)^{\frac{q}{qR+N(q-1)}} \\ &+ C\rho^{-\frac{N+q}{(q-1)(R+N+1)}} \left( \int_{t-\rho^\kappa}^t \int_{B_\rho} u^R \right)^{\frac{1}{R+N+1}} + C\rho^{-\frac{N+q}{R+1-q}} \left( \int_{t-\rho^\kappa}^t \int_{B_\rho} u^R \right)^{\frac{1}{R+1-q}}, \end{aligned}$$

where  $C = C(N, q, R)$ . Now from estimate (3.3) of Lemma 3.2,

$$\begin{aligned} \sup_{B_{\eta/2}} u(., t) &\leq C\rho^{-\frac{\kappa N}{qR+N(q-1)}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_\eta)})^{\frac{Rq}{qR+N(q-1)}} \\ &+ C\rho^{-\frac{N+q}{(q-1)(R+N+1)} + \frac{\kappa}{R+N+1}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_\eta)})^{\frac{R}{R+N+1}} \\ &+ C\rho^{-\frac{-(N+q)+\kappa}{R+1-q}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_\eta)})^{\frac{R}{R+1-q}}. \end{aligned}$$

Let  $\tau < T$ , and  $k_0 \in \mathbb{N}$  such that  $k_0\eta^\kappa/2 \geq \tau$ . For any  $t \in (0, \tau]$ , there exists  $k \in \mathbb{N}$  with  $k \leq k_0$  such that  $t \in (k\eta^\kappa/2, (k+1)\eta^\kappa/2]$ . taking  $\rho^\kappa = t/(k+1)$ , we find for any  $0 < t < \tau$ , and  $C = C(N, q, R)$ ,

$$\begin{aligned} \sup_{B_{\eta/2}} u(., t) &\leq C \left( \frac{1 + \eta^{-\kappa}\tau}{t} \right)^{\frac{N}{qR+N(q-1)}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_\eta)})^{\frac{Rq}{qR+N(q-1)}} \\ &+ C \left( \frac{1 + \eta^{-\kappa}\tau}{t} \right)^{\frac{N+q}{R+N+1} - \frac{\kappa(q-1)-1}{R+N+1}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_\eta)})^{\frac{R}{R+N+1}} \\ &+ C \left( \frac{1 + \eta^{-\kappa}\tau}{t} \right)^{\frac{N+q-1}{R+1-q} - \frac{\kappa}{R+1-q}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_\eta)})^{\frac{R}{R+1-q}}. \end{aligned} \quad (6.10)$$

If we choose  $\kappa$  such that  $\kappa\varepsilon(N+q)q' \geq 1$ , we obtain, with  $C = C(N, q, R, \eta, \varepsilon, \tau)$ ,

$$\begin{aligned} \sup_{B_{\eta/2}} u(., t) &\leq Ct^{-\frac{N}{qR+N(q-1)}} (t + \|u_0\|_{L^R(B_\eta)})^{\frac{Rq}{qR+N(q-1)}} \\ &+ Ct^{\frac{1-\varepsilon}{R+N+1}} (t + \|u_0\|_{L^R(B_\eta)})^{\frac{R}{R+N+1}} + Ct^{\frac{1-\varepsilon}{R+1-q}} (t + \|u_0\|_{L^R(B_\eta)})^{\frac{R}{R+1-q}} \end{aligned} \quad (6.11)$$

And in fact the second term can be absorbed by the first one, with a new constant depending on  $\tau$ , and we finally obtain (1.15). ■

**Remark 6.3** These estimate in  $t^{-N/(qR+N(q-1))}$  improves the estimate in  $t^{-N/2R}$  of the first regularizing effect when  $q > q_*$ . And it appears to be sharp. Indeed consider for example the particular solutions given in [25] of the form  $u_C(x, t) = Ct^{-a/2}f(|x|/\sqrt{t})$ , where  $\eta \mapsto f(\eta)$  is bounded,  $f'(0) = 0$  and  $\lim_{\eta \rightarrow \infty} \eta^a f(\eta) = C$ . Then  $u_C$  is solution of (1.1) in  $Q_{\mathbb{R}^N \setminus \{0\}, \infty}$ , with initial data  $C|x|^{-a}$ . When  $a < N$ , that means  $q > q_*$ , then  $|x|^{-a} \in L_{loc}^R(\mathbb{R}^N)$  for any  $R \in [1, N/a)$ , and  $u_C$  is solution in  $Q_{\mathbb{R}^N, \infty}$ . We have  $\sup_{B_1} u(., t) = Cf(0)t^{-a/2}$ . Taking  $N/R = a(1 + \delta)$ , for small  $\delta > 0$  our estimate near  $t = 0$  gives  $\sup_{B_1} u(., t) \leq C_\delta t^{-\frac{a}{2}(1+\delta)}$ .

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